

Spacetime Emergence Through A Geometric Renormalization Method

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In collaboration with
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The Method in a Nutshell

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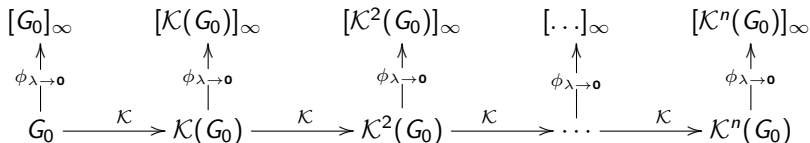
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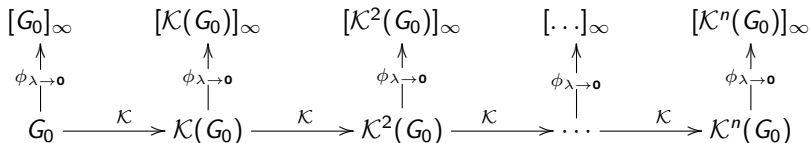
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- end in a fixed point, a set of accumulation points or converge under \mathcal{K} (non-generic scenario).

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Given metric spaces $X, Y \in \mathcal{S}$, a map $f : X \rightarrow Y$, where $\exists \lambda \geq 1, \epsilon \geq 0$ such that $\forall x_1, x_2 \in X$

$$\frac{1}{\lambda} d_X(x_1, x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \epsilon, \quad \forall x_1, x_2 \in X$$

and

$$\forall y \in Y : \exists x \in X : d_Y(y, f(x)) \leq C.$$

is a quasi-isometry, i.e.

- distance of the images under f , within a factor λ , and up to a constant, of their original distances, and
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k -local insertion/deletion of edges: the k -neighborhood of vertices of a graph G such that the k -neighborhood remains invariant. Pure quasi-isometry.

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$G \longrightarrow \mathcal{C}(G)$ “clique graph”:

- Vertices $V(\mathcal{C}(G))$: maximal subsimplexes (cliques) of G ,
- An edge between two of vertices of $\mathcal{C}(G)$ if cliques have non-zero vertex overlap (in G).

Rough isometry.

A Measure of Similarity (and Convergence): d_{GH}

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- \mathcal{K} is pure quasi-isometry, then $d_{GH}(X, Y) = \infty$. Spaces are structurally different.
- \mathcal{K} is rough isometry, then $d_{GH}(X, Y) = \text{finite}$. Spaces are, to some extent, structurally similar.

Rescaling ϕ_λ and Continuum Limit Properties

The rescaling map, ϕ_λ , on a metric space (X, d_X)

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λ parametrizes the distance between the points on the different length scales.

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- $\lim_{\lambda \rightarrow \infty} \phi_\lambda$ reveals the fine structure of X by magnifying the infinitesimal neighborhoods of the points of X ,
- $\lim_{\lambda \rightarrow 0} \phi_\lambda$ corresponds to the large scale structure of X :

$$\lim_{\lambda \rightarrow \infty} \phi_\lambda ((X, d_X)) = \lim_{\lambda \rightarrow \infty} (X, \lambda d_X) = (X_\infty, d_{X,\infty}),$$

important for us.

Some Properties of the Continuum Limit

If spaces are purely quasi-isometric, i.e. $d_{GH}(X, Y) = \infty$, then

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i.e. they also have different continuum limits.

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Set $\left\{ (X', d_{X'}) \mid d_{GH}(X', X) < \infty \right\}$ is the basin of attraction, for the attractor $(X_\infty, d_{X,\infty})$, under the evolution map ϕ_λ . They all have the same unique continuum limit $(X_\infty, d_{X,\infty})$.

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$(X_\infty, d_{X,\infty})$ is scale invariant under ϕ_λ i.e. $d_{GH}(X_\infty, \lambda X_\infty) = 0$.

The Full Picture

Combination of two operations

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- However, they are homeomorphic; can even be chosen to be the same topological space.
- Implies that different levels of spacetime will have different metric even if they are the same set.

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The Full Picture

Combination of two operations

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with $G_i = \mathcal{K}^i(G_0)$, etc.

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 - ❷ These roughly isometric spaces are in the basin of attraction for the corresponding continuum limit (optimally the classical space-time).
- ❷ coarse graining chain reaches a stable fixed point/set of accumulation points
 - ❶ if the spaces are uniformly compact, the Gromov's compactness theorem shows they convergence with respect to d_{GH} . Not generic.

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For graphs with polynomial growth, dimension

$$D(G) = \lim_{r \rightarrow \infty} \frac{\log \beta(G, v_i, r)}{\log r}$$

coincides with usual dim for embedded spaces and lattices.

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- Graphs with locally finite vertex degree, being connected, and vertex transitive: integer dimension.
- D stable under quasi-isometry \mathcal{K} .
- If we want the change of D under renormalization: not quasi-isometric \mathcal{K} but translocal.

Summary

- A geometric RG: using notions of isometric coarse graining, and rescaling.
- Coarse graining goes on until a phase transition occurs, quasi \rightarrow rough isometry.
- Quasi-isometric spaces are structurally different, have different continuum limits.
- Rough isometric spaces are structurally similar, have the same unique continuum limit.
- Dimension stable under k -local RG.
- Change of dimension under RG with translocal operations.

Rescaling ϕ_λ and continuum limit properties

Take lattice \mathbb{Z}^n embedded in \mathbb{R}^n , take the scaling limit

$$\phi_l : (\mathbb{Z}^n, d_{\mathbb{Z}^n}) \longmapsto (\mathbb{Z}^n, \lambda d_{\mathbb{Z}^n}), \quad \lambda = 2^{-l}$$

with $d_{\mathbb{Z}^n}$ a suitable metric on \mathbb{Z}^n . Then

$$\lim_{\lambda \rightarrow 0} (\mathbb{Z}^n, \lambda d_{\mathbb{Z}^n}) = \mathbb{R}^n,$$

in pointed GH-sense. For a fixed ball around $x = 0$, and for $l \rightarrow \infty$ the ball is more and more filled with points stemming from lattices having edge length 2^{-l} .