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A Mode-Sum Scheme for Renormalized Stress-Energy Tensors in Arbitrary Dimensions

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Renormalization

- “Removing infinities” by reabsorbing them into redefinitions of the theory’s parameters.
- Ubiquitous in both classical and quantum field theories.
- Examples:

1. Radiation reaction in classical electrodynamics:

$$(A_a, m) \rightarrow (\hat{A}_a, \hat{m})$$

2. Semi-classical gravity:

$$R_{ab} - \frac{1}{2}g_{ab} + \alpha A_{ab} + \beta B_{ab} + \dots = 8\pi \langle T_{ab} \rangle$$

\downarrow

$$(\alpha, \beta, \dots) \rightarrow (\hat{\alpha}, \hat{\beta}, \dots) \longleftarrow \langle T_{ab} \rangle_{\text{ren}}$$

$$\langle H | \phi^2 | H \rangle_{\text{ren}} = \lim_{x \rightarrow x'} \left[G_E(x, x') - G_S(x, x') \right]$$

- $G_E(x, x')$ is typically only known in terms of its mode expansion. Global information is encoded in the boundary conditions on the mode functions.

$$G_E(x, x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} e^{in\kappa\Delta\tau} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \gamma) g_{nl}(r, r')$$

- $G_S(x, x')$ is defined only in terms of a local geometrical expansion.

$$G_S(x, x') = \frac{1}{8\pi^2} \left(\frac{\Delta^{1/2}}{\sigma} + V \log \sigma / \ell^2 \right)$$

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Problem: How do we subtract these in such a way that a meaningful limit can be taken?

Candelas Howard Approach

Simplify by taking partial coincidence limits $\mathbf{x} \rightarrow \mathbf{x}'$

$$\begin{aligned} G_S &\sim \frac{1}{g_{\tau\tau}\Delta\tau^2} + O(1) \\ &= -\frac{\kappa^2}{g_{\tau\tau}} \sum_{n=1}^{\infty} n \cos n\kappa\Delta\tau + O(1). \end{aligned}$$

$$\langle |H\phi^2|H \rangle_{\text{ren}} = \frac{1}{(4\pi M)^2} \sum_{n=1}^{\infty} \left\{ \sum_{l=0}^{\infty} \left[(2l+1)g_{nl}(r,r) - \frac{M}{r f^{1/2}} \right] + \frac{n}{2f} \right\} + O(1)$$

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Is there a more direct approach?

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Expand Hadamard parametrix in these variables, e.g.

$$G_S = \frac{1}{8\pi^2} \left(\frac{2}{s^2} + \left[A(r) + B(r) \frac{w^2}{s^2} + C(r) \frac{w^4}{s^4} \right] + \dots \right)$$

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Any order of convergence can be obtained decomposing higher order terms:

$$g_{nl}^{(2i-2)}(r) = \frac{2^{2i} i! \Gamma(i + \frac{1}{2})}{\sqrt{\pi} \kappa^{2i} r^2} \sum_{j=0}^i S_{ij}(r) \frac{(-1)^{n+j}}{2^{j+1} \kappa^{2j} r^{2j} j!} \left(\frac{1}{\eta} \frac{\partial}{\partial \eta} \right)^j \left\{ \sum_{k=0}^{i-|n|} \frac{P_l^{-k}(\eta) Q_l^k(\eta)}{(i-k-|n|)!(i+k+|n|)!} \right. \\ \left. + \sum_{k=\max\{1, |n|-i\}}^{|n|+i} \frac{P_l^{-k}(\eta) Q_l^k(\eta)}{(i+k-|n|)!(i-k+|n|)!} \right\}$$

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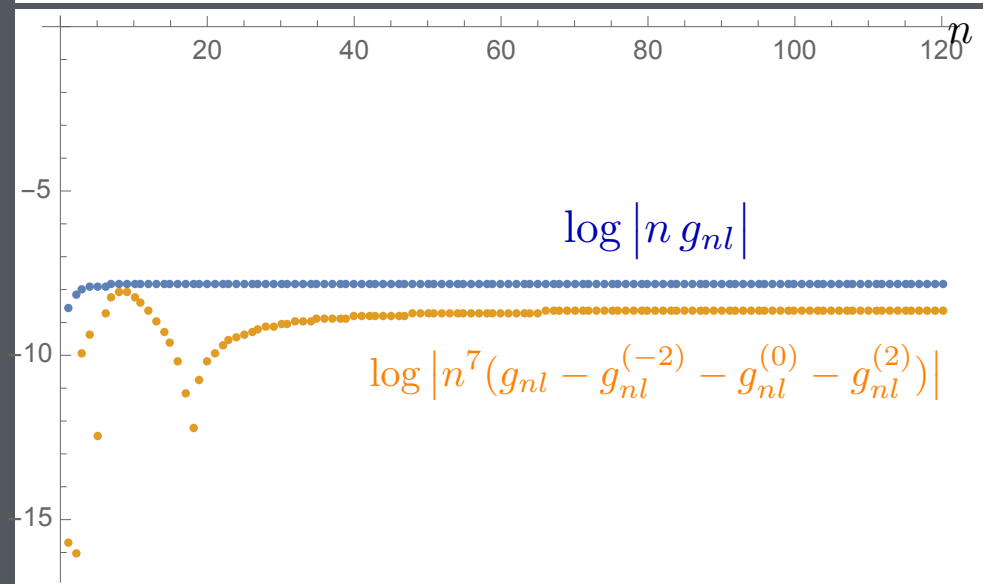
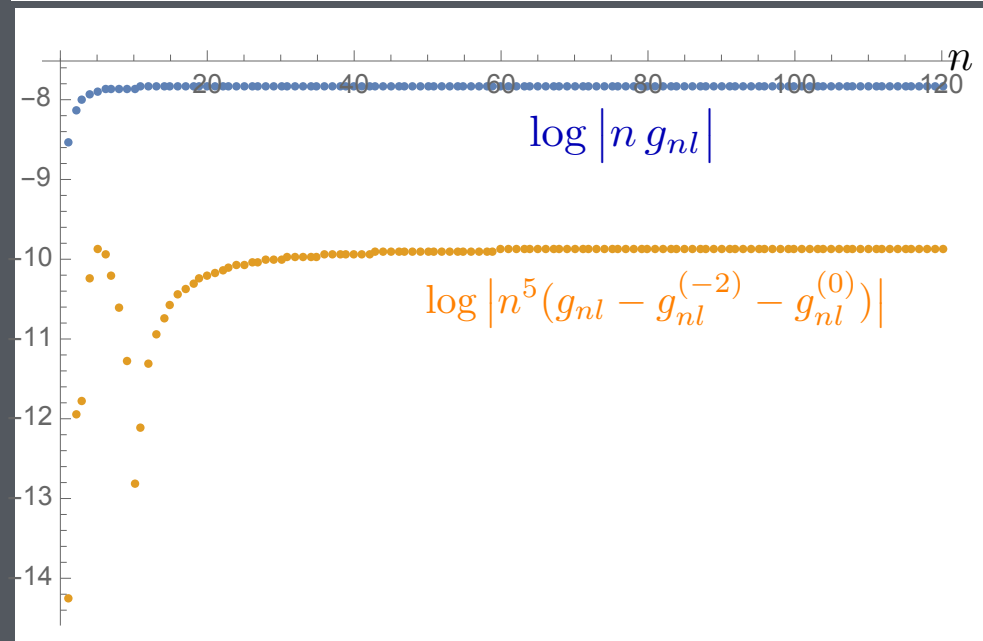
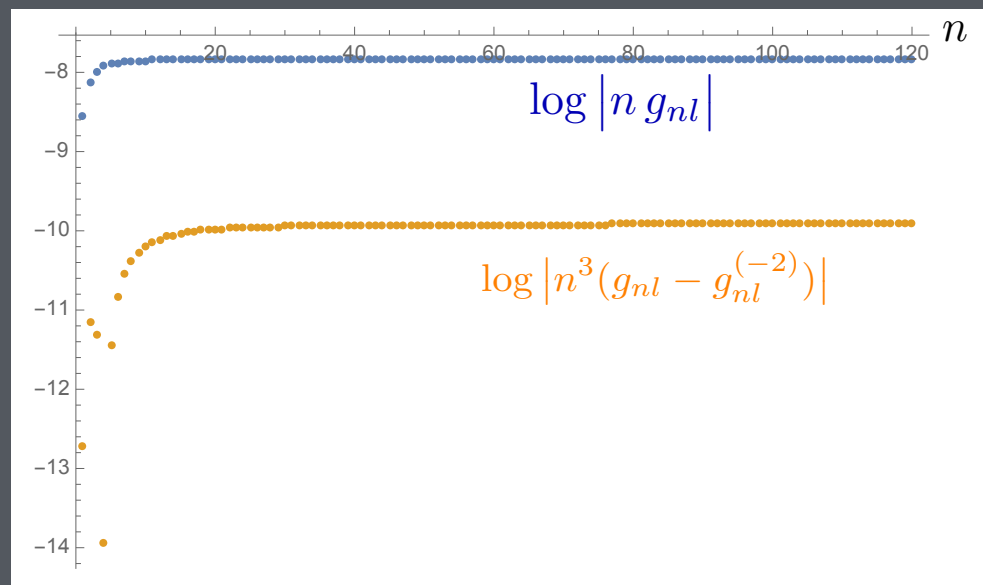
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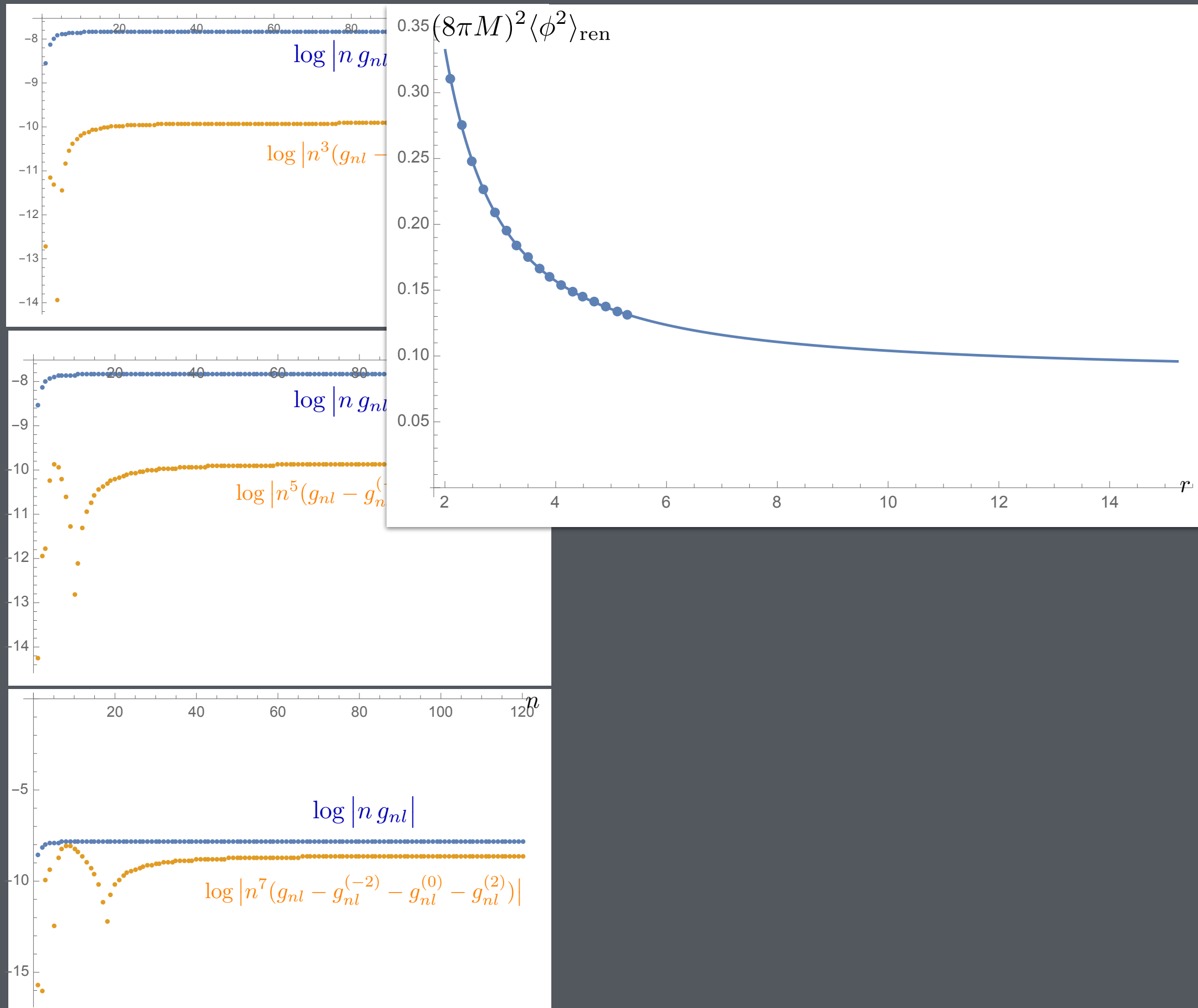
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$$\begin{array}{c} \propto n^{-2i-3} \\ \propto l^{-2i-2} \end{array}$$

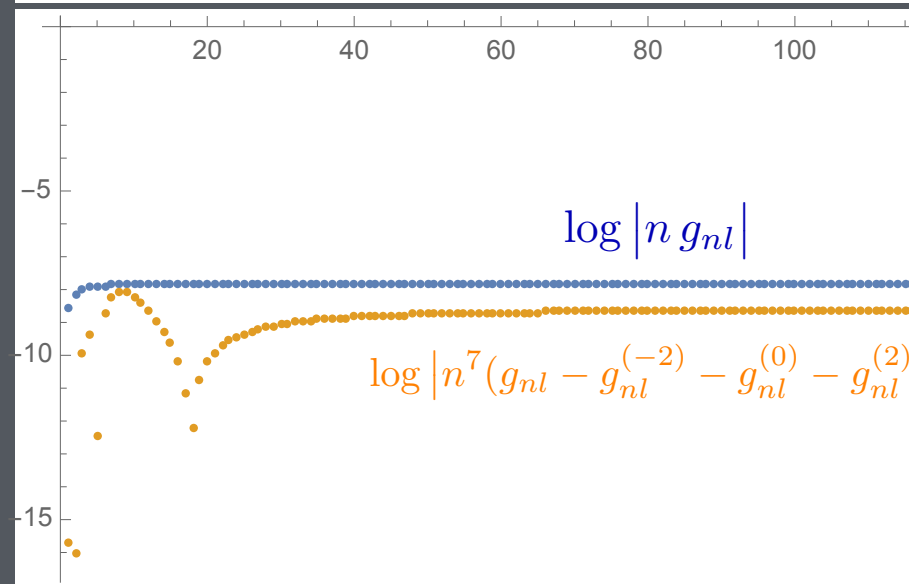
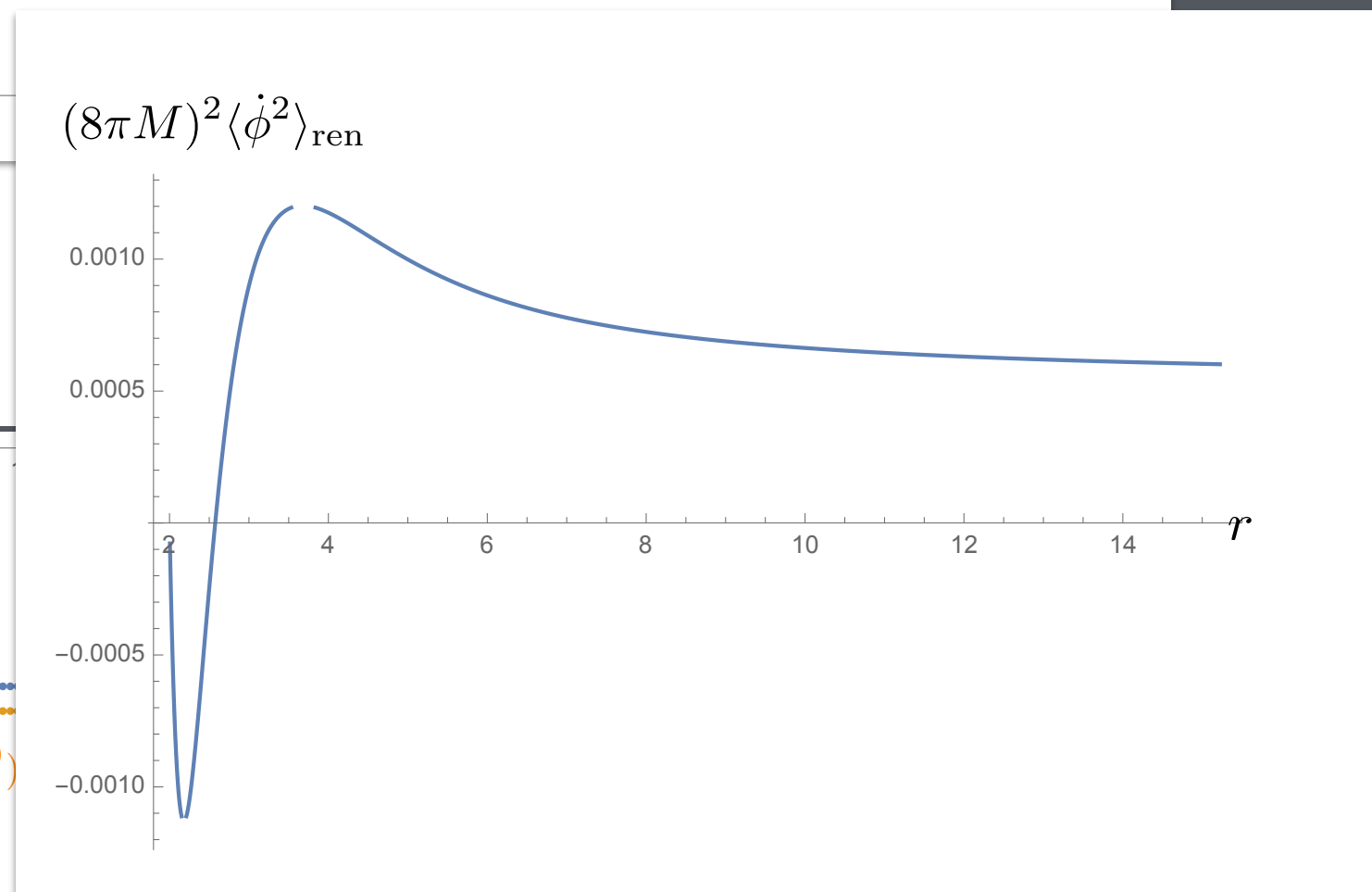
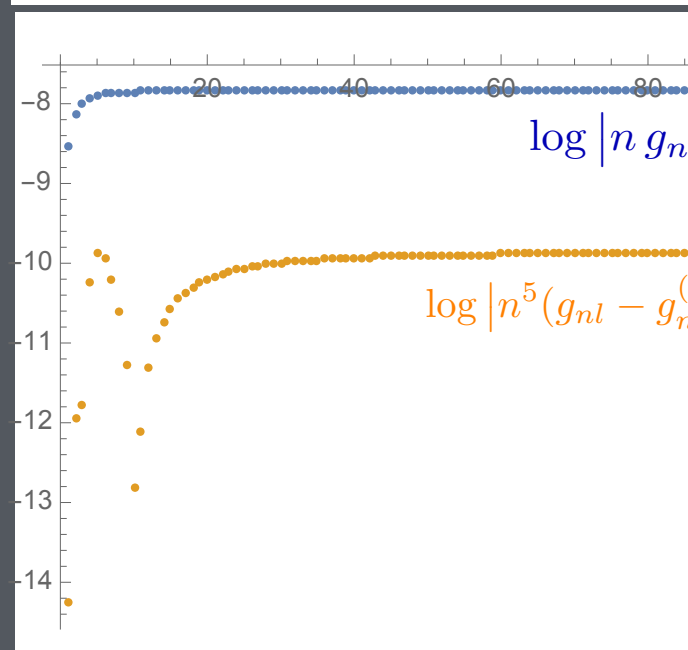
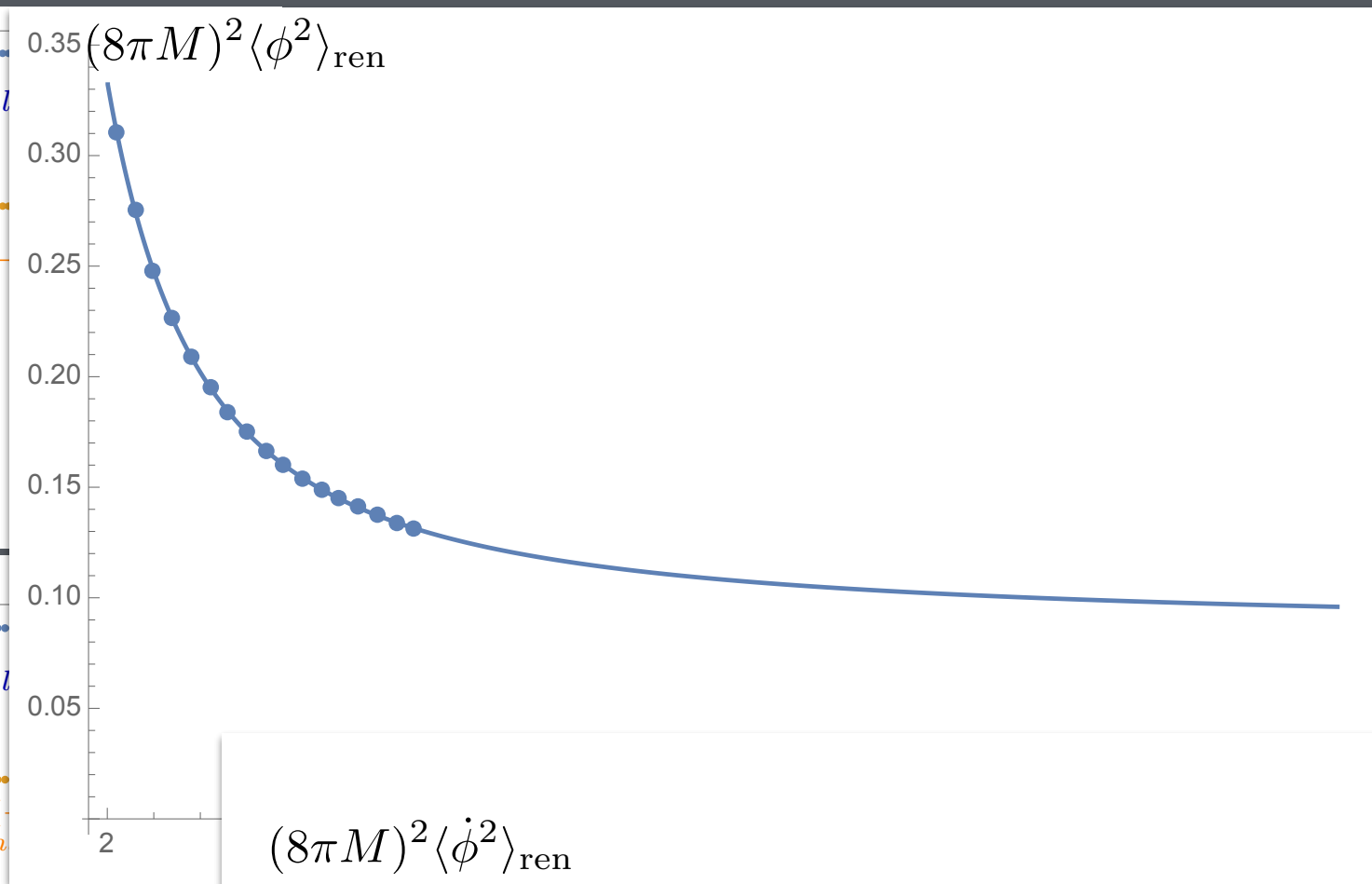
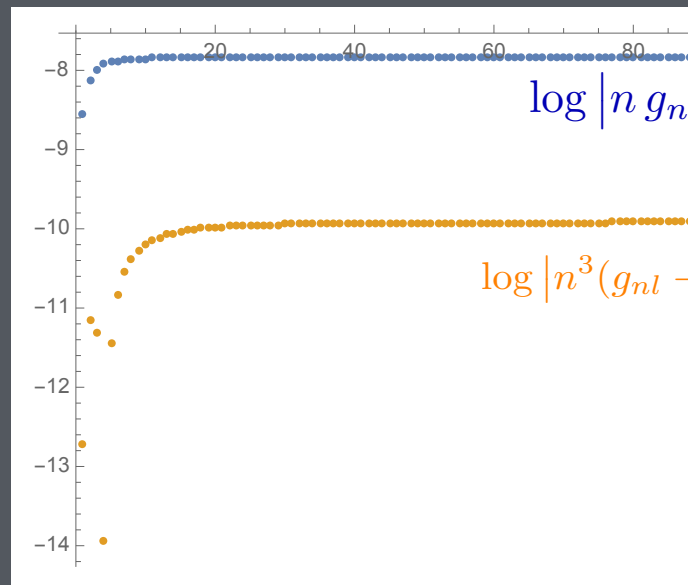
Convergence Plots



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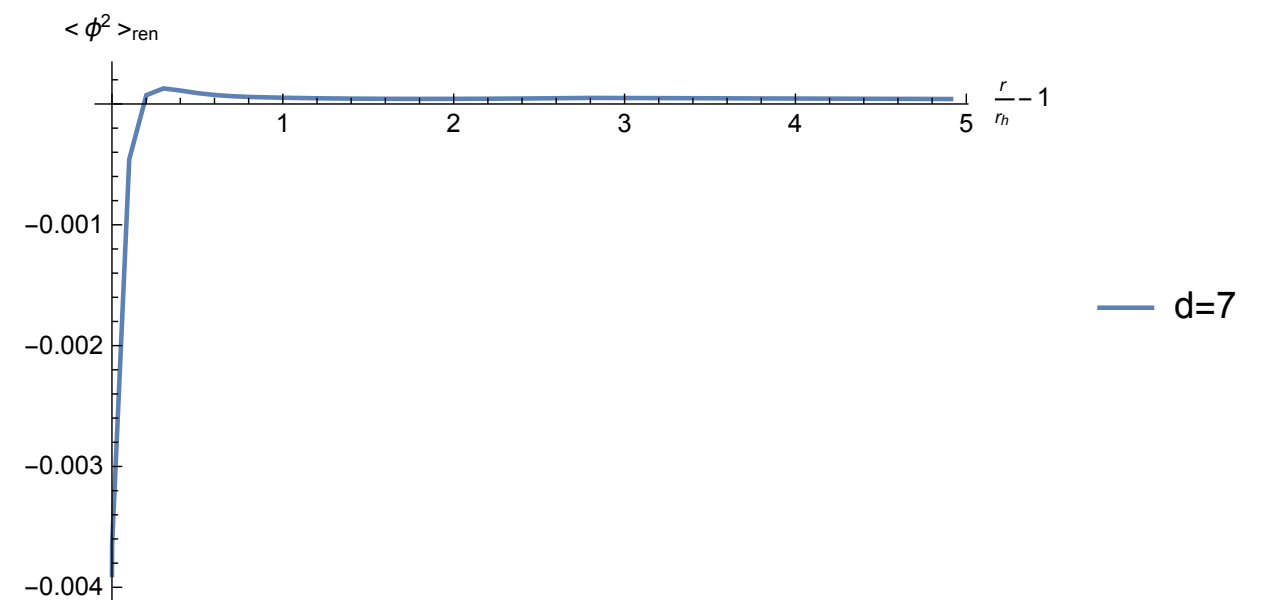
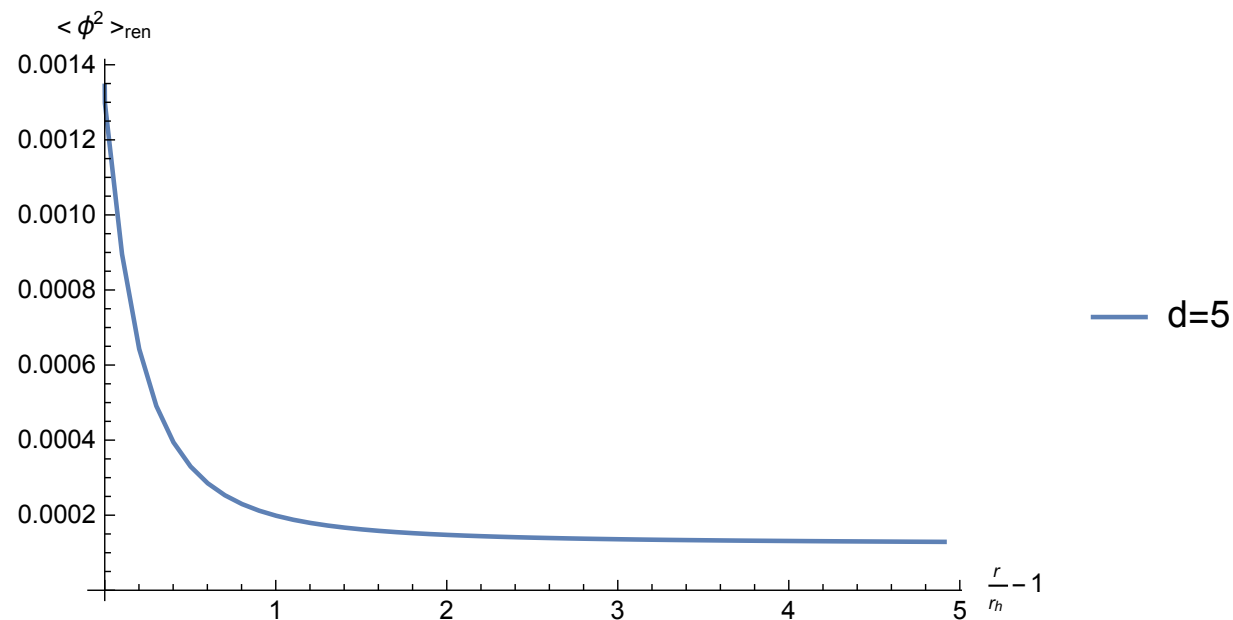
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Vacuum Polarization in Higher D

Regularization parameters in closed form can also be obtained to any order in arbitrary dimensions. For example in odd D, we have

$$g_{nl}^{(2i-2)}(r) = \frac{2^{2i} i! \Gamma(i + \frac{1}{2})}{\sqrt{\pi} \kappa^{2i} r^{d-2}} \sum_{j=0}^i S_{ij}(r) \frac{(-1)^{n+j}}{2^{j+1} \kappa^{2j} r^{2j} \Gamma(j + \frac{d}{2} - 1)} \left(\frac{1}{\eta} \frac{\partial}{\partial \eta} \right)^j \left\{ \sum_{k=0}^{i-|n|} \frac{P_{l+\frac{d}{2}-2}^{-k}(\eta) Q_{l+\frac{d}{2}-2}^k(\eta)}{(i-k-|n|)!(i+k+|n|)!} \right. \\ \left. + \sum_{k=\max\{1, |n|-i\}}^{|n|+i} \frac{P_{l+\frac{d}{2}-2}^{-k}(\eta) Q_{l+\frac{d}{2}-2}^k(\eta)}{(i+k-|n|)!(i-k+|n|)!} \right\}$$



Conclusions

- Splitting in multiple directions guarantees a mode-by-mode subtraction.
- A judicious choice of expansion “coordinates” permits us compute all regularization parameters in closed form.
- High order regularization parameters gives rapidly convergent mode-sums.
- No residual divergences arise.
- Resultant mode-sums are uniformly convergent, even close to the horizon.
- Regularization is surprisingly agnostic to number of dimensions.