

The Spacetime Between Einstein and Kaluza-Klein

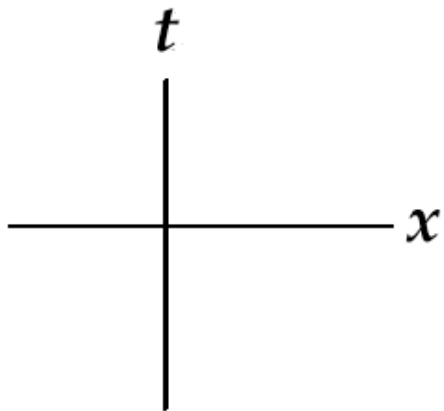
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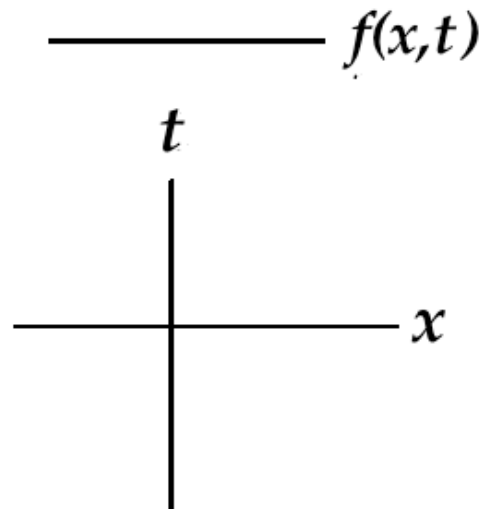
Daytona Beach, Florida

Three Space-Times

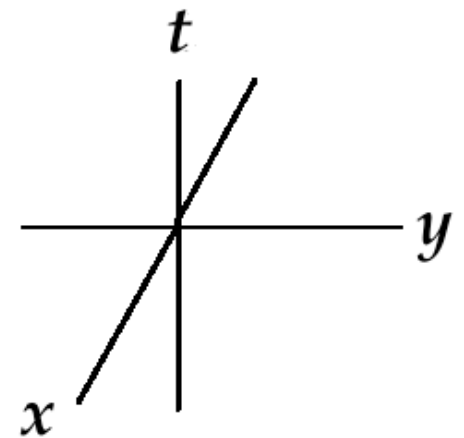
Einstein



In Between Spacetime



Kaluza-Klein



Constructing the Spacetime

- Create direct sums of a scalar field and 4-vector and 4-covector fields.
- Take all tensor products of such fields, creating a subalgebra of the universal covering algebra of the tensor algebra.
- Write a general Lagrangian up to rank two, vary it with respect to the 4-velocity.
- Read off the equivalent Cristoffel symbols and calculate the analogous curvatures.

Step 1: Create the Subalgebra of the Universal Covering Algebra

Define the “scalar-extended” x-vectors and x-covectors:

$$P^B = P^0 \oplus P^b \quad \text{and} \quad Q_A = Q_0 \oplus Q_a \quad (1)$$

and all tensor products of these x-vectors and x-covectors. Because P_0 and Q_0 are scalar fields, coordinate transformations are effected via, for example,

$$\bar{P}^A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial \bar{x}^a}{\partial x^b} \end{pmatrix} P^B \quad (2)$$

Step 2: Define a second order element of the subalgebra, an extended momentum x-vector, and evaluate it.

$$\tilde{g}_{AB} = \psi \oplus A_a \oplus A_b \oplus g_{ab} \quad (4)$$

$$P^A = e \oplus p^a \quad (5)$$

$$\tilde{g}(P^A, P^B) = e^2 \psi + e A_a p^a + e A_b p^b + g_{ab} p^a p^b \quad (6)$$

Step 3: Vary \tilde{g} with respect to the four-velocity:

$$\begin{aligned}
 & \frac{dv^f}{d\lambda} + \frac{1}{2}g^{df} \left(\frac{\partial g_{db}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d} \right) v^a v^b \\
 & + \frac{1}{2m} \frac{e}{m} g^{df} \left(\frac{\partial A_d}{\partial x^a} - \frac{\partial A_a}{\partial x^d} \right) v^a + \\
 & + \frac{1}{2m} \frac{e}{m} g^{df} \left(\frac{\partial A_d}{\partial x^b} - \frac{\partial A_b}{\partial x^d} \right) v^b - \frac{1}{2m^2} \frac{e^2}{m} g^{df} \frac{\partial \psi}{\partial x^d} = 0
 \end{aligned}
 \tag{7}$$

Step 4: Define the extended derivative operator and covariant derivative and write the directional derivative.

Extended derivative operator:

$$\frac{\partial}{\partial \tilde{x}^A} = \begin{cases} 0 & \text{if } A = 0 \\ \frac{\partial}{\partial x^a} & \text{if } A = a \end{cases} \quad (8)$$

The directional derivative:

$$V^A \tilde{\nabla}_A V^F = V^A \frac{\partial V^F}{\partial \tilde{x}^A} + V^A \tilde{\Gamma}_{AB}^F V^B \quad (9)$$

where $\tilde{\nabla}_A$ is the extended covariant derivative.

Step 5: Compare to generalized derivative to the variation of Step 3 and identify non-zero equivalent Cristoffel and Cristoffel-like symbols! (Here, the index 0 is a scalar index, whereas roman indices in italics are vector indices.)

$$\frac{dv^f}{d\lambda} + \frac{e^2}{m^2} \tilde{\Gamma}_{00}^F + \frac{e}{m} v^a \tilde{\Gamma}_{a0}^F + \frac{e}{m} v^b \tilde{\Gamma}_{0b}^F + \tilde{\Gamma}_{ab}^F v^a v^b = 0$$

$$\tilde{\Gamma}_{00}^0 = \tilde{\Gamma}_{a0}^0 = \tilde{\Gamma}_{0b}^0 = \tilde{\Gamma}_{ab}^0 = 0$$

$$\tilde{\Gamma}_{00}^f = -\frac{1}{2} g^{df} \frac{\partial \psi}{\partial x^d}$$


$$\tilde{\Gamma}_{a0}^f = \frac{1}{2} g^{df} \left(\frac{\partial A_d}{\partial x^a} - \frac{\partial A_a}{\partial x^d} \right)$$

$$\tilde{\Gamma}_{0b}^f = \frac{1}{2} g^{df} \left(\frac{\partial A_d}{\partial x^b} - \frac{\partial A_b}{\partial x^d} \right)$$

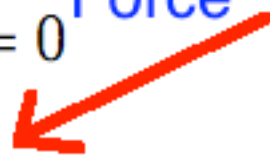
$$\tilde{\Gamma}_{ab}^f = \Gamma_{ab}^f$$

$$\frac{dv^f}{d\lambda} + \frac{e^2}{m^2} \tilde{\Gamma}_{00}^F + \frac{e}{m} v^a \tilde{\Gamma}_{a0}^F + \frac{e}{m} v^b \tilde{\Gamma}_{0b}^F + \tilde{\Gamma}_{ab}^F v^a v^b$$

pseudo-Newtonian


acceleration 

$$\tilde{\Gamma}_{00}^0 = \tilde{\Gamma}_{a0}^0 = \tilde{\Gamma}_{0b}^0 = \tilde{\Gamma}_{ab}^0 = 0$$


Force 

$$\tilde{\Gamma}_{00}^f = -\frac{\alpha}{2} g^{df} \frac{\partial \psi}{\partial x^d}$$

$$\tilde{\Gamma}_{a0}^f = \frac{1}{2} g^{df} \left(\frac{\partial A_d}{\partial x^a} - \frac{\partial A_a}{\partial x^d} \right)$$

electromagnetic forces 

$$\tilde{\Gamma}_{0b}^f = \frac{1}{2} g^{df} \left(\frac{\partial A_d}{\partial x^b} - \frac{\partial A_b}{\partial x^d} \right)$$

gravity force 

$$\tilde{\Gamma}_{ab}^f = \Gamma_{ab}^f$$

Constraints Reducing Kaluza-Klein C-Symbols to the In-Between Theory

$$\nabla_a \psi A^a = 0$$

$$\frac{\partial \psi}{\partial x^d} - A^a F_{da} = 0$$

$$\frac{1}{2} \left(\frac{\partial A_d}{\partial x^a} + \frac{\partial A_a}{\partial x^d} \right) - A_b \Gamma_{ad}^b = 0$$

These constraints are required to get the correct particle Lagrangian from 5-d Kaluza-Klein—probably showing why that theory won't give the right particle dynamics.

Step 6: Derive the equivalent Ricci tensor

$$\tilde{R}_{AB} = \frac{\partial \tilde{\Gamma}_{AB}^C}{\partial \tilde{x}^C} - \frac{\partial \tilde{\Gamma}_{AC}^C}{\partial \tilde{x}^B} + \tilde{\Gamma}_{AB}^E \tilde{\Gamma}_{CE}^C - \tilde{\Gamma}_{AC}^E \tilde{\Gamma}_{BE}^C \quad (10)$$

Note: This x-tensor can be derived in the usual way by a commutation of successive x-covariant derivatives operating on an x-vector. The result is of some interest!

Step 7: Obtain the components of the Ricci x-tensor

$$\tilde{R}_{ab} = \frac{\partial \Gamma_{ab}^c}{\partial x^c} - \frac{\partial \Gamma_{ac}^c}{\partial x^b} + \Gamma_{ab}^e \Gamma_{ce}^c - \Gamma_{ac}^e \Gamma_{be}^c \quad (11)$$

$$\tilde{R}_{a0} = \frac{1}{2} \left(\frac{\partial F_a^c}{\partial x^c} + \Gamma_{cd}^c F_a^d - \Gamma_{ca}^d F_d^c \right) \quad (12)$$

with

$$F_{ba} = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} \quad (13)$$

$$\tilde{R}_{00} = -\frac{\alpha}{2} \left[\frac{\partial}{\partial x^c} \left(g^{dc} \frac{\partial \psi}{\partial x^d} \right) + \Gamma_{ce}^c \left(g^{de} \frac{\partial \psi}{\partial x^d} \right) \right] \quad (14)$$

$$-\frac{1}{4} g^{de} g^{fc} F_{dc} F_{fe}$$

$$\tilde{R}_{ab} = \frac{\partial \Gamma_{ab}^c}{\partial x^c} - \frac{\partial \Gamma_{ac}^b}{\partial x^b} + \Gamma_{ab}^e \Gamma_{ce}^c - \Gamma_{ac}^e \Gamma_{be}^c \quad (11)$$

GR in Vacuo!

$$\tilde{R}_{a0} = \frac{1}{2} \left(\frac{\partial F_a^c}{\partial x^c} + \Gamma_{cd}^c F_a^d - \Gamma_{ca}^d F_c^d \right) \quad (12)$$

with

E&M in Vac!

$$F_{ba} = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} \quad (13)$$

$$\begin{aligned} \tilde{R}_{00} = & -\frac{\alpha}{2} \left[\frac{\partial}{\partial x^c} \left(g^{dc} \frac{\partial \psi}{\partial x^d} \right) + \Gamma_{ce}^c \left(g^{de} \frac{\partial \psi}{\partial x^d} \right) \right] \\ & - \frac{1}{4} g^{de} g^{fc} F_{dc} F_{fe} \end{aligned} \quad (14)$$

Klein-Gordon Equation!

Electromagnetic
Energy Density!

Step 8: Define the Extended Stress-Energy

$$\tilde{T}^{AB} = \begin{pmatrix} \bar{\alpha}\rho & \beta J^b \\ \beta J^a & \kappa T^{ab} \end{pmatrix} \quad (15)$$

where $\bar{\alpha}$, β , and κ are constants, and ρ is the energy density, which may be presumed to be proportional to the number density.

Step 9: Variational Derivation of the Field Equations

The most natural field Lagrangian is

$$\mathcal{L} = \int \tilde{R} \sqrt{-g} \, d^4x \quad (16)$$

where \tilde{R} is the analogue of the Ricci scalar. Note that this field Lagrangian is over a four-manifold. The variation then yields the following field equations for the nonlinear metric:

$$\tilde{R}_{AB} - \frac{1}{2}\tilde{R}g_{ab} + V_{AB} = \kappa\tilde{T}_{AB} \quad (17)$$

$$V_{AB} = \Omega \begin{pmatrix} \psi A^e B_e & \frac{1}{2} (\psi \delta_b^e + A^e A_b) B_e \\ \frac{1}{2} (\psi \delta_a^e + A^e A_a) B_e & \delta_a^e A_b B_e \end{pmatrix} \quad (18)$$

where

$$B_e = \nabla^f F_{ef} \quad (19)$$

and

$$\Omega = \left(\psi - A_d A^d \right)^{-1} \quad (20)$$

These equations give GR, Maxwell's equations, and a Klein-Gordon field in vacuum, the latter with an unusual dependence on the electric energy density. Inside matter, Maxwell's equations become nonlinear.

Step 10: An elementary cosmological solution to the field equations

$$\tilde{R}_{00} - \frac{1}{2} \left(\psi^{-1} \tilde{R}_{00} + R \right) \psi = \bar{\alpha} \rho \quad (21)$$

$$R_{ab} - \frac{1}{2} \left(\psi^{-1} \tilde{R}_{00} + R \right) g_{ab} = \kappa T_{ab} \quad (22)$$

These two equations can be decoupled to the following two equations:

$$\tilde{R}_{00} = \left(\frac{2}{3} \bar{\alpha} \rho - \frac{1}{3} \kappa T \right) \psi \quad (23)$$

and

$$R_{ab} - R g_{ab} = \kappa T_{ab} + \bar{\alpha} \rho g_{ab} \quad (24)$$

$$ds^2 = -d\tau^2 + a^2(\tau) \left(dx^2 + dy^2 + dz^2 \right) \oplus \psi(\tau) \quad (25)$$

Step 11: The Solution

$$d\tilde{s}^2 = \left(-d\tau^2 + \tau^{10/9}(dx^2 + dy^2 + dz^2) \right) \oplus (a\tau^{n_+} + b\tau^{n_-}) \quad (26)$$

$$a(\tau) = \tau^{5/9} \quad (27)$$

$$\psi = a\tau^{n_+} + b\tau^{n_-} \quad (28)$$

$$n_+ = -\frac{1}{3} + \sqrt{\frac{1}{9} + \beta} \quad (29)$$

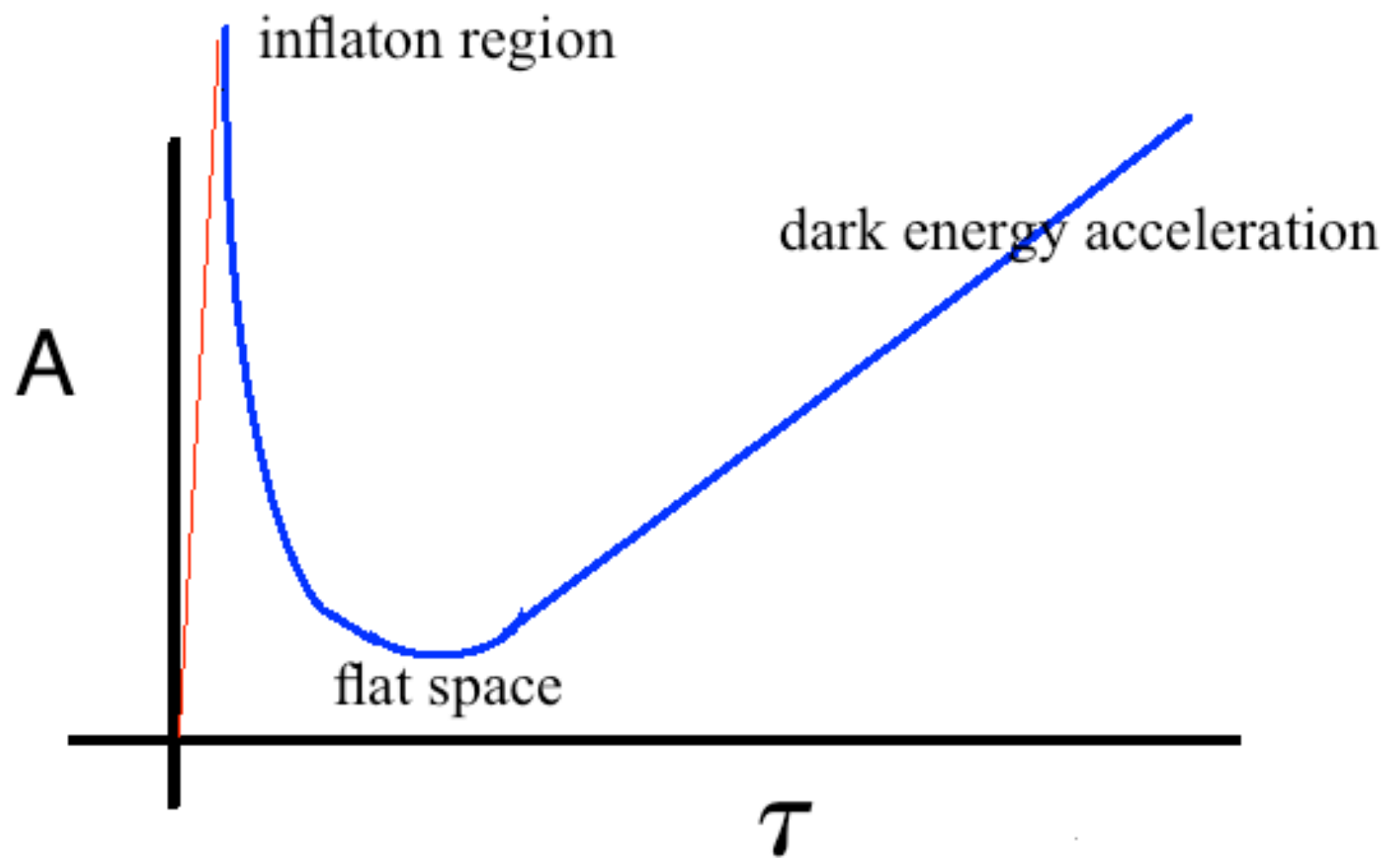
and

$$n_- = -\frac{1}{3} - \sqrt{\frac{1}{9} + \beta} \quad (30)$$

Effective expansion function $A(v, \tau)$:

$$A(v, \tau) = \sqrt{v^2\tau^{10/9} + a\tau^{n_+} + b\tau^{n_-}} \quad (31)$$

Many possible cosmologies, including dark energy cosmologies and inflaton-like “pre-expanded” cosmologies.



Scalar Potential May Provide Extra Attraction

$$\begin{aligned}
 & -\frac{\alpha}{2} \left[\frac{\partial}{\partial x^c} \left(g^{dc} \frac{\partial \psi}{\partial x^d} \right) + \Gamma_{ce}^c \left(g^{de} \frac{\partial \psi}{\partial x^d} \right) \right] - \frac{1}{4} g^{de} g^{fc} F_{dc} F_{fe} \\
 & \quad \quad \quad = \left(\frac{2}{3} \bar{\alpha} \rho - \frac{1}{3} \kappa T \right) \psi
 \end{aligned}$$

The diagram illustrates the components of the equation above, with red arrows pointing from labels to specific terms:

- Scalar Field Terms** (blue text) points to the first term: $-\frac{\alpha}{2} \left[\frac{\partial}{\partial x^c} \left(g^{dc} \frac{\partial \psi}{\partial x^d} \right) + \Gamma_{ce}^c \left(g^{de} \frac{\partial \psi}{\partial x^d} \right) \right]$.
- Galactic Magnetic Field** (blue text) points to the second term: $-\frac{1}{4} g^{de} g^{fc} F_{dc} F_{fe}$.
- Stress-Energy Terms** (blue text) points to the right-hand side of the equation: $\left(\frac{2}{3} \bar{\alpha} \rho - \frac{1}{3} \kappa T \right) \psi$.