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Does the Gauss-Bonnet term stabilize wormholes?

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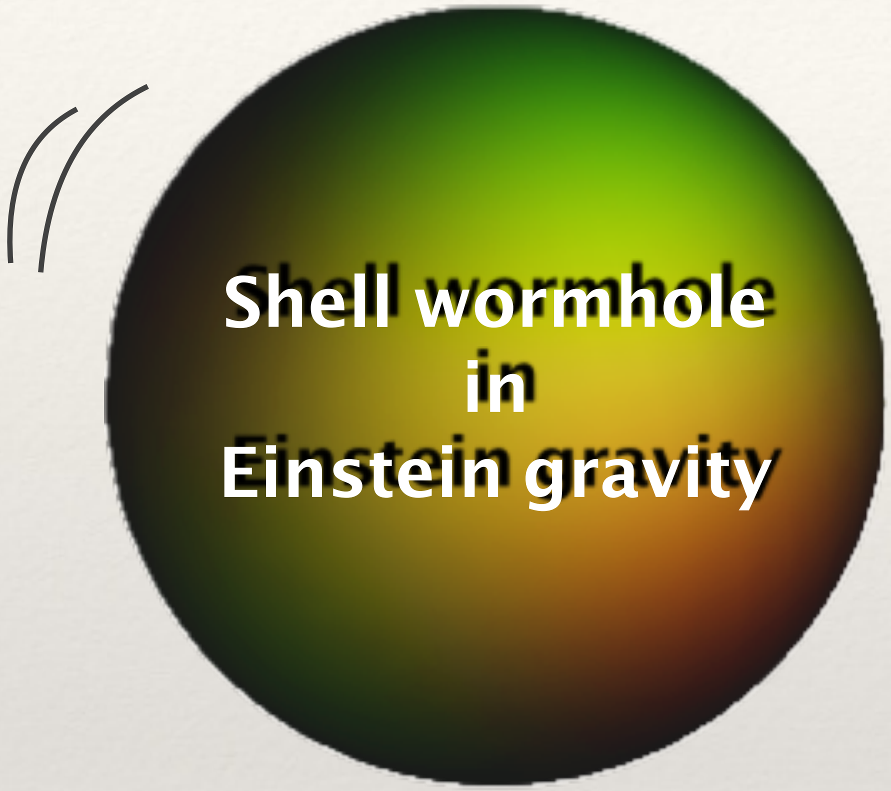


- **STABILITY against RADIAL PERTURBATIONS.**
- **Comparing two wormholes (ANALYTICALLY).**
  → Revealing the effect of the Gauss-Bonnet term on Stability.

Introduction

Traversable wormholes are fascinating objects.
→ space-time short cut, time travel.

Problem: instability, use of exotic matter, great tidal force…

→Stability of wormhole spacetime is the first priority.

Einstein-Gauss-Bonnet gravity

$$S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}(R - 2\lambda + \alpha L_{GB}) \qquad L_{\mathrm{GB}} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

The Gauss-Bonnet term appears in the action as the ghost-free quadratic curvature correction term in the low-energy limit of heterotic superstring theory in ten dimensions.

Einstein gravity

$$S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}(R-2\Lambda)$$

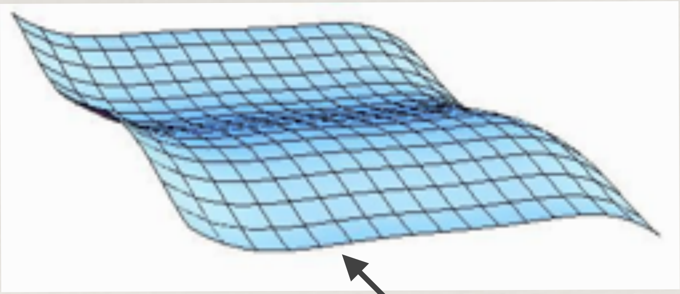
Vacuum solutions

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\Omega^k_{d-2})^2, \qquad f(r) = k - \frac{m}{r^{d-3}} - \tilde{\Lambda}r^2$$

$$\left(\tilde{\Lambda} := \frac{2\Lambda}{(d-1)(d-2)},\ k = \pm 1, 0\right)$$

Construction

Assumption for symmetry: Z_2 **symmetry**



$$\mathrm{d}s^2_{d-1} = -\mathrm{d}\tau^2 + a^2(\tau)(\mathrm{d}\Omega^k_{d-2})^2$$

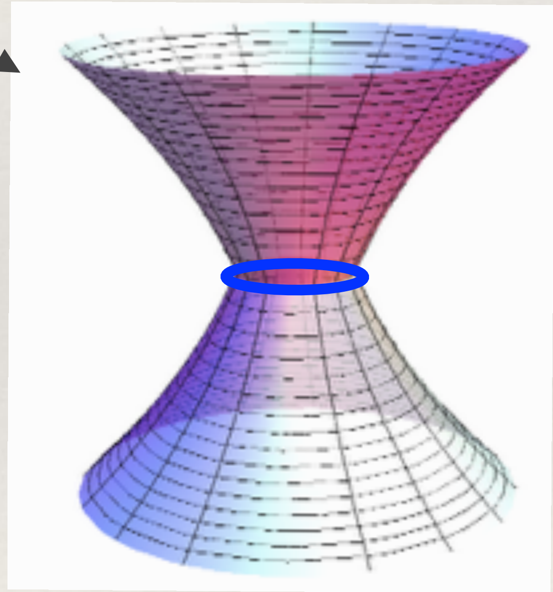
Junction conditions

$$[K^i{}_j]_\pm - \delta^i{}_j[K]_\pm = -8\pi S^i{}_j$$

$$\downarrow$$

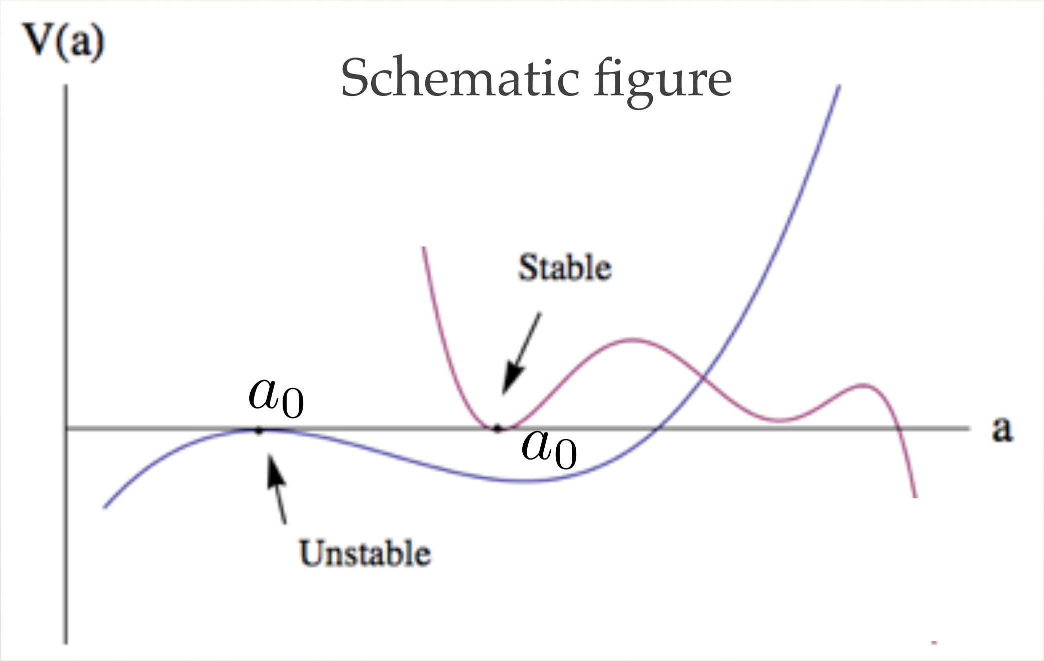
$$\sigma = -\frac{d-2}{4\pi a}\sqrt{f+\dot{a}^2},$$

$$p = \frac{1}{4\pi}\left(\frac{\ddot{a}+\frac{1}{2}f'}{\sqrt{f+\dot{a}^2}} + \frac{d-3}{a}\sqrt{f+\dot{a}^2}\right)$$



Assumption for matter: **negative tension**

Master equation & Stability



Birkhoff's theorem $\rightarrow$
no gravitational waves from radial motion

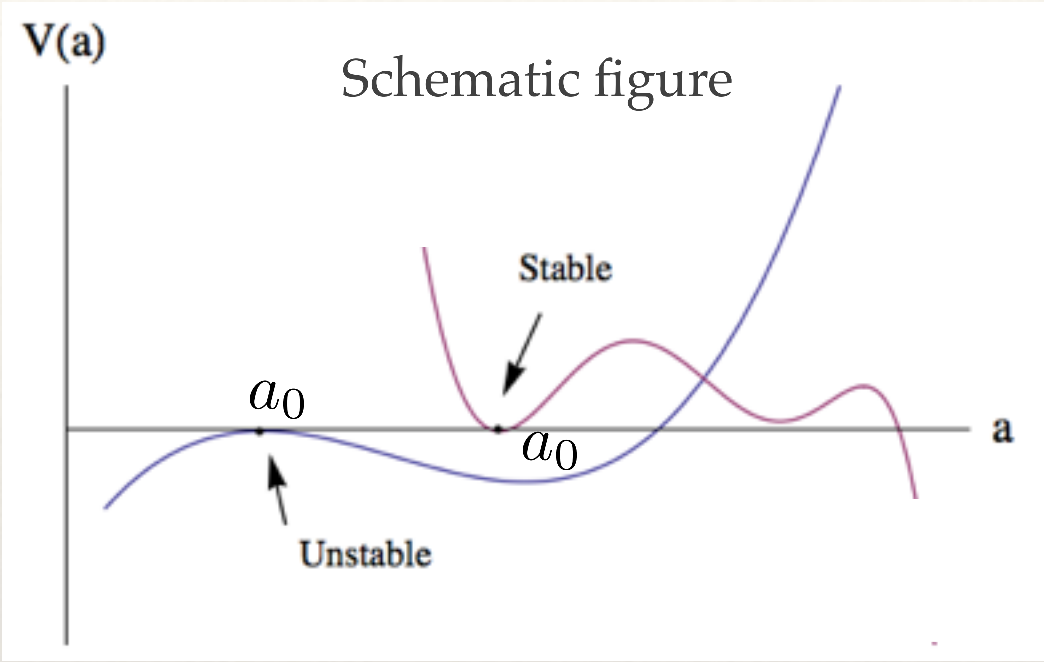
Master eq for radial motion : $\dot{a}^2 + V(a) = 0,$

$$V(a) = f(a) - \left(\frac{4\pi\sigma}{d-2}\right)^2 a^2$$

Stable: $V''(a_0) > 0$

Unstable: $V''(a_0) < 0$

Master equation & Stability



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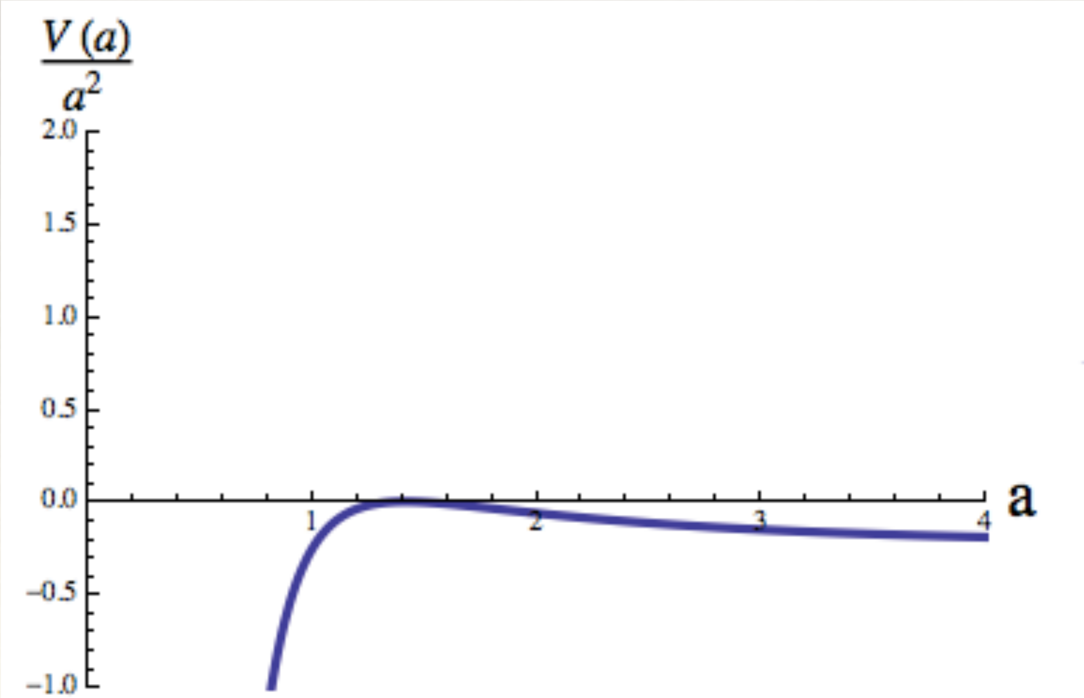
Stable: $V''(a_0) > 0$

Unstable: $V''(a_0) < 0$

For k=1,m>0 : $\boxed{V''(a_0) = -\frac{2(d-3)k}{a_0^2}}$

Spherical wormhole
is unstable



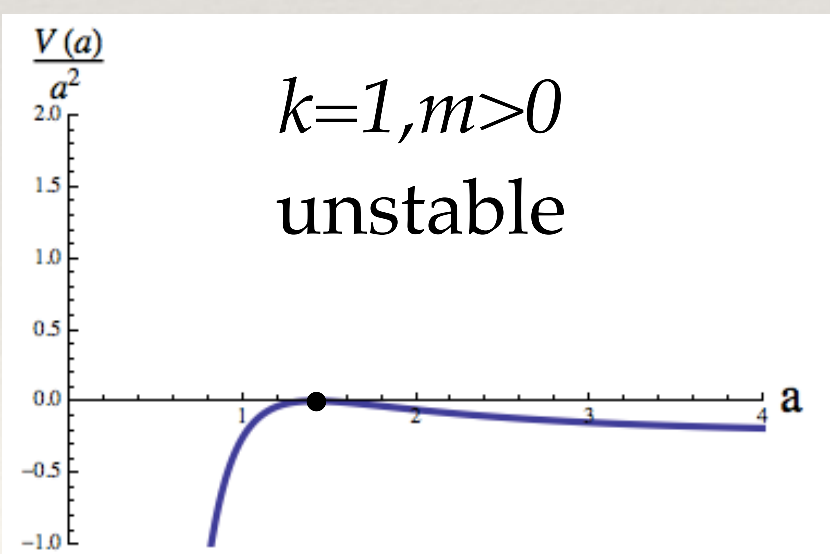
※Generalization of Einstein-Maxwell system→ T.K, Harada, 2015

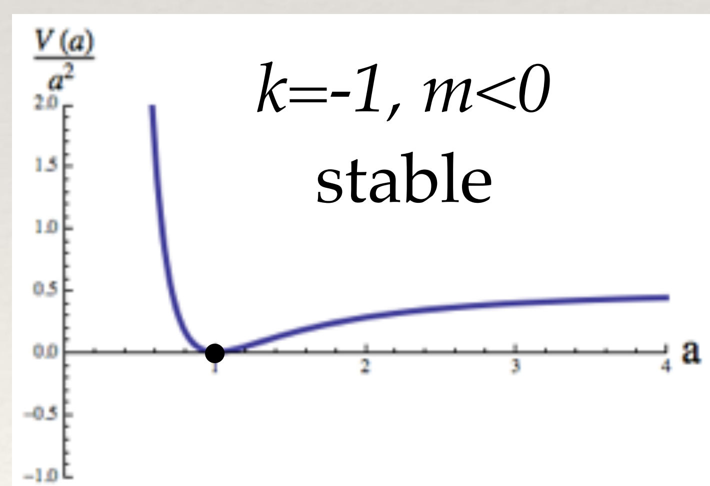
Classification

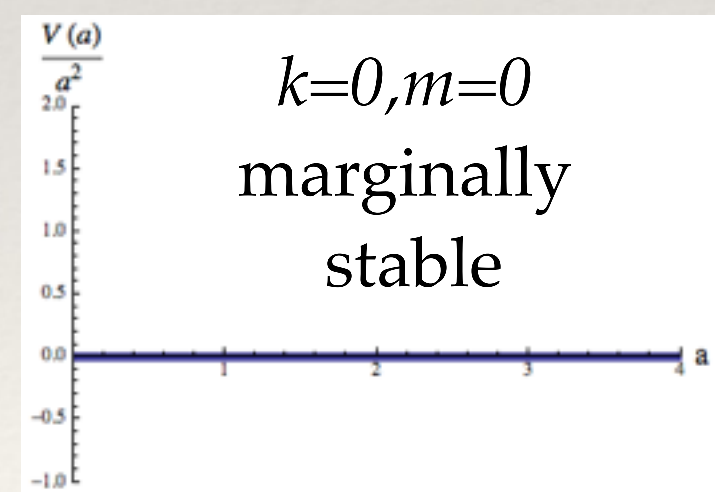
$$a_c^{(\mathrm{GR})} := \left(\frac{(d-3)k}{(d-1)\tilde{\Lambda}}\right)^{1/2}.$$

$$m_c^{(\mathrm{GR})} := \frac{2k}{d-1}\left(\frac{(d-3)k}{(d-1)\tilde{\Lambda}}\right)^{(d-3)/2}$$

		Existence	Possible range of a_0	Stability
$k=1$	$\Lambda > 0$	$0 < m < m_c^{(\mathrm{GR})}$	$0 < a_0 < a_c^{(\mathrm{GR})}$	Unstable
	$\Lambda \leq 0$	$m > 0$	$a_0 > 0$	Unstable
$k=0$	$\Lambda \geq 0$	None	–	–
	$\Lambda < 0$	$m = 0$	$a_0 > 0$	Marginally stable
$k=-1$	$\Lambda \geq 0$	None	–	–
	$\Lambda < 0$	$m < m_c^{(\mathrm{GR})} (< 0)$	$a_0 > a_c^{(\mathrm{GR})}$	Stable







Einstein-Gauss-Bonnet gravity

$$\boxed{S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}(R - 2\lambda + \alpha L_{GB})} \qquad (d \geq 5)$$

$$L_{\mathrm{GB}} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

α : coupling constant, inverse string tension

$$\boxed{\alpha > 0}$$

Vacuum solution

$$\mathrm{d}s_d^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\gamma_{AB}\mathrm{d}z^A\mathrm{d}z^B$$

$$f(r) := k + \frac{r^2}{2\tilde{\alpha}}\left(1 \mp \sqrt{1 + \frac{4\tilde{\alpha}m}{r^{d-1}} + 4\tilde{\alpha}\tilde{\Lambda}}\right)$$

$$\tilde{\Lambda} := \frac{2\Lambda}{(d-1)(d-2)}$$

$$\tilde{\alpha} := (d-3)(d-4)\alpha$$

$-$: the GR branch
$+$: non-GR branch

in GR branch, $\alpha \to 0$

$$f(a) = k - \frac{m}{r^{d-3}} - \tilde{\Lambda}r^2$$

GR branch

Construction

Junction conditions: $[K^i{}_j]_\pm - \delta^i{}_j[K]_\pm + 2\alpha\Big(3[J^i{}_j]_\pm - \delta^i{}_j[J]_\pm - 2P^i{}_{kjl}[K^{kl}]_\pm\Big) = -\kappa_d^2 S^i{}_j$

$$J_{ij} := \frac{1}{3}\left(2KK_{ik}K^k{}_j + K_{kl}K^{kl}K_{ij} - 2K_{ik}K^{kl}K_{lj} - K^2K_{ij}\right), \quad P_{ikjl} := \mathcal{R}_{ikjl} + 2h_{i[l}\mathcal{R}_{j]k} + 2h_{k[j}\mathcal{R}_{l]i} + \mathcal{R}h_{i[j}h_{l}$$

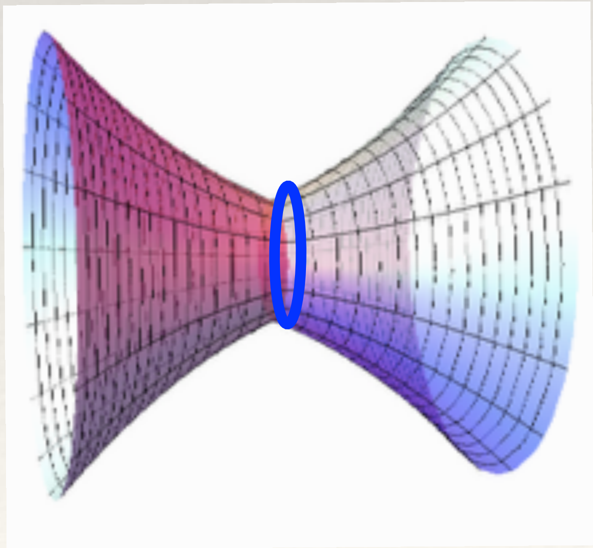
$$\downarrow$$

$$4\pi G\rho(a) = -\frac{(d-2)\sqrt{f+\dot a^2}}{a}\left\{1 + \frac{2\tilde\alpha}{3}\left(2\frac{\dot a^2}{a^2} + \frac{3k}{a^2} - \frac{f}{a^2}\right)\right\},$$

$$-4\pi G p(a) = -\frac{a}{\sqrt{f+\dot a^2}}\left\{\frac{\ddot a}{a} + \frac{f'}{2a} + (d-3)\left(\frac{\dot a^2}{a^2} + \frac{f}{a^2}\right)\right\} - \frac{2\tilde\alpha a}{\sqrt{f+\dot a^2}}\left\{\frac{d-5}{3}\left(\frac{\dot a^2}{a^2} + \frac{f}{a^2}\right)\left(2\frac{\dot a^2}{a^2} + \frac{3k}{a^2} - \frac{f}{a^2}\right)\right.$$
$$\left. + \left(2\frac{\dot a^2}{a^2} + \frac{k}{a^2} + \frac{f}{a^2}\right)\frac{\ddot a}{a} + \frac{f'}{2a}\left(\frac{k}{a^2} - \frac{f}{a^2}\right)\right\}$$

Assumption for matter: **negative tension**

Assumption for symmetry: Z_2 **symmetry**



Master equation& Stability analysis

Birkhoff's theorem → no gravitational waves from radial motion

Master eq. for radial motion : $\dot{a}^2 + V(a) = 0$

$$V(a) := f(a) - J(a)a^2,$$

$$J(a) := \frac{\left(B(a) - A(a)^{1/2}\right)^2}{4\tilde{\alpha}B(a)}, \quad B(a) := \left\{18\tilde{\alpha}\Omega^2 + A(a)^{3/2} + 6\sqrt{\tilde{\alpha}\Omega^2(9\tilde{\alpha}\Omega^2 + A(a)^{3/2})}\right\}^{1/3}. \quad \Omega := \frac{16\pi^2\sigma^2}{(d-2)^2}$$

Stability criterion

$$V''(a_0) = -\frac{2kP(a_0)}{a_0^2(a_0^2 + 2k\tilde{\alpha} + 2\tilde{\alpha}f_0)(a_0^2 + 2k\tilde{\alpha} - 2\tilde{\alpha}f_0)},$$

$$P(a_0) := 4\tilde{\alpha}^2 f_0\left\{6k - (d-3)f_0\right\} + (a_0^2 + 2k\tilde{\alpha})\left\{(d-3)a_0^2 + 2(d-5)k\tilde{\alpha}\right\}$$

$$\longrightarrow \quad \boxed{V''(a_0) \propto -kP(a_0)}$$

$k = \pm 1, 0$

Instability for $k = 1$ with $m > 0$

$$\boxed{V''(a_0) \propto -P(a_0)}$$

$$
\begin{aligned}
\mathbf{P}(\mathbf{a_0}) =& 4\tilde{\alpha}^2 f_0 \left\{ 6 - (d-3) f_0 \right\} \\
& + (a_0^2 + 2\tilde{\alpha}) \left\{ (d-3)a_0^2 + 2(d-5)\tilde{\alpha} \right\} \\
>& 4\tilde{\alpha}^2 f_0 \left\{ 6 - (d-3) f_0 \right\} + 2\tilde{\alpha} f_0 \left\{ (d-3)a_0^2 + 2(d-5)\tilde{\alpha} \right\} \\
=& 2\tilde{\alpha} f_0 \left\{ 2(d-3)\tilde{\alpha} \left(\frac{a_0^2}{2\tilde{\alpha}} - f_0 \right) + 2(d+1)\tilde{\alpha} \right\} \\
>& 2\tilde{\alpha} f_0 \left\{ -2(d-3)\tilde{\alpha} + 2(d+1)\tilde{\alpha} \right\} = 16\tilde{\alpha}^2 f_0 > \mathbf{0}.
\end{aligned}
$$

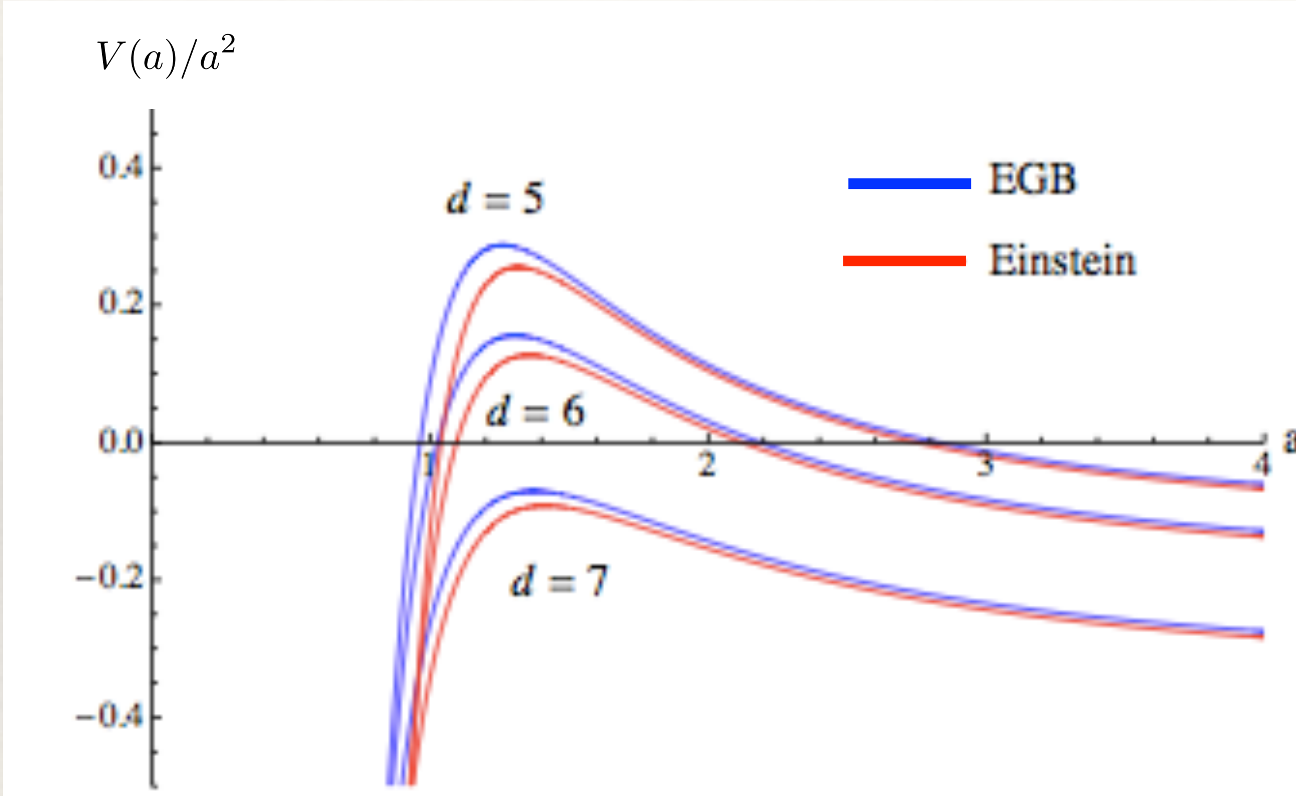


Figure: The potential $\bar{V}(a)$ for $d = 5, 6, 7$ in Einstein and Einstein-Gauss-Bonnet (EGB) gravity with $k = 1$, $\alpha = 0.02$, $m = 1$, $\Lambda = 1$ and $\sigma = -0.1$.

$\rightarrow$ $\boxed{P(a_0) > 0}$

Spherical wormhole is unstable

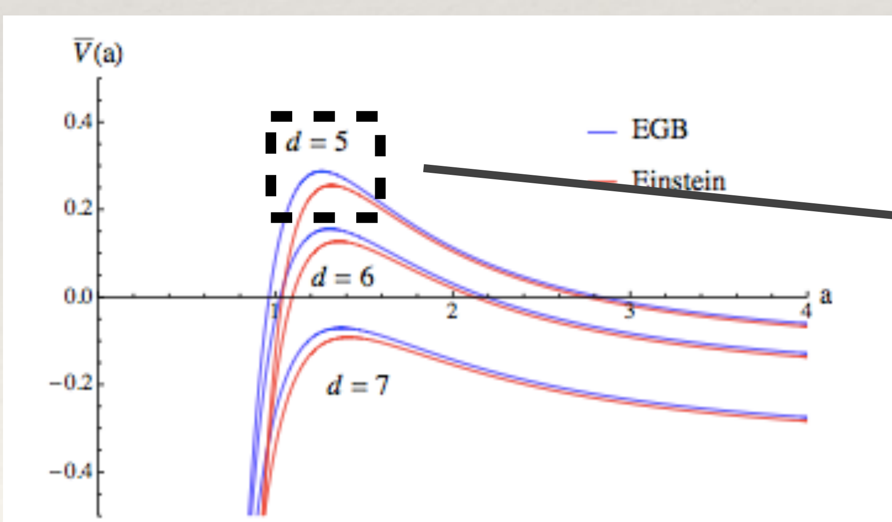
Effect of the Gauss-Bonnet term on Stability

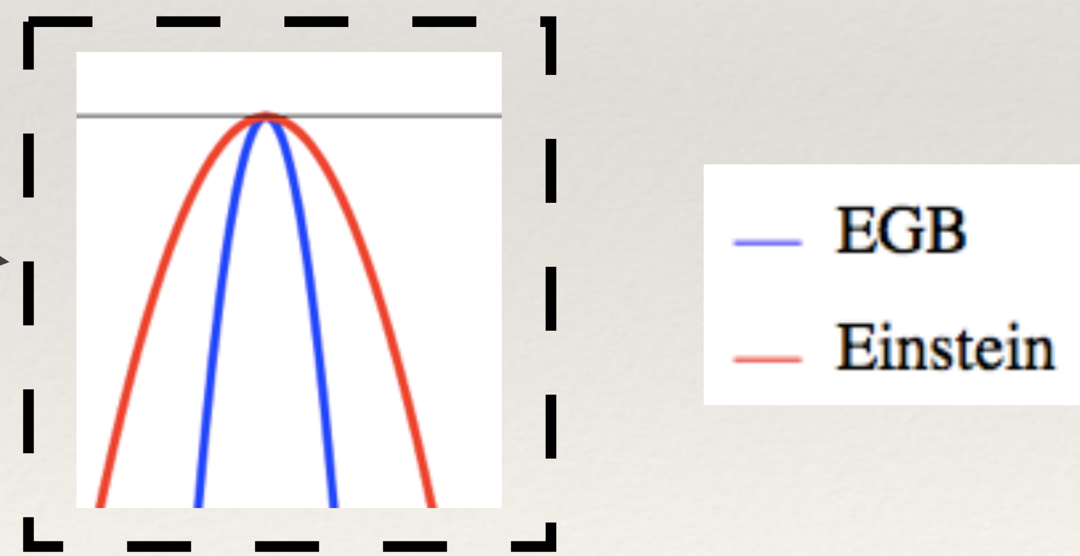
GB shell wormhole turned out to be unstable.
But, how unstable is it when compared with Einstein shell wormhole?

sufficiently small α : $\quad \epsilon := \tilde{\alpha}/a_{\mathrm{E}}^2 \ll 1$

$\rightarrow$ Perturbative analysis : $a_0 = a_{\mathrm{E}} + a_{(1)}\epsilon + a_{(2)}\epsilon^2 + \ldots$.

$\rightarrow$ ϵ expansion up to 1st order : $V''_{\mathrm{GB}}(a_0) \simeq V''_{\mathrm{Einstein}}(a_{\mathrm{E}}) - k\dfrac{8 f_{\mathrm{E}}(a_{\mathrm{E}})}{a_{\mathrm{E}}^2}\epsilon$





GB-potential is Steeper than Einstein-potential.

Classification

		Static solutions exist?	Stability
$k=1$	$m>0$	Yes	Unstable
	$m \leq 0$	No	–
$k=0$	$m=0$	$\Lambda \geq 0$: No	–
		$\Lambda < 0$: Yes	Marginally Stable
	$m \neq 0$	No	–
$k=-1$	$m \geq 0$	No	–
	$m<0$	$\Lambda \geq 0$: No	–
		$-(2d-5)/(2d-1) < 4\tilde{\alpha}\tilde{\Lambda} < 0$: Yes	Stable
		$4\tilde{\alpha}\tilde{\Lambda} = -(2d-5)/(2d-1)$: Yes	Stable or Marginally Stable
		$-1 < 4\tilde{\alpha}\tilde{\Lambda} < -(2d-5)/(2d-1)$ with $d=5$: Yes	Stable or Marginally Stable
		$-1 < 4\tilde{\alpha}\tilde{\Lambda} < -(2d-5)/(2d-1)$ with $d \geq 6$: Yes	Stable, Marginally Stable or Unstable

Summary

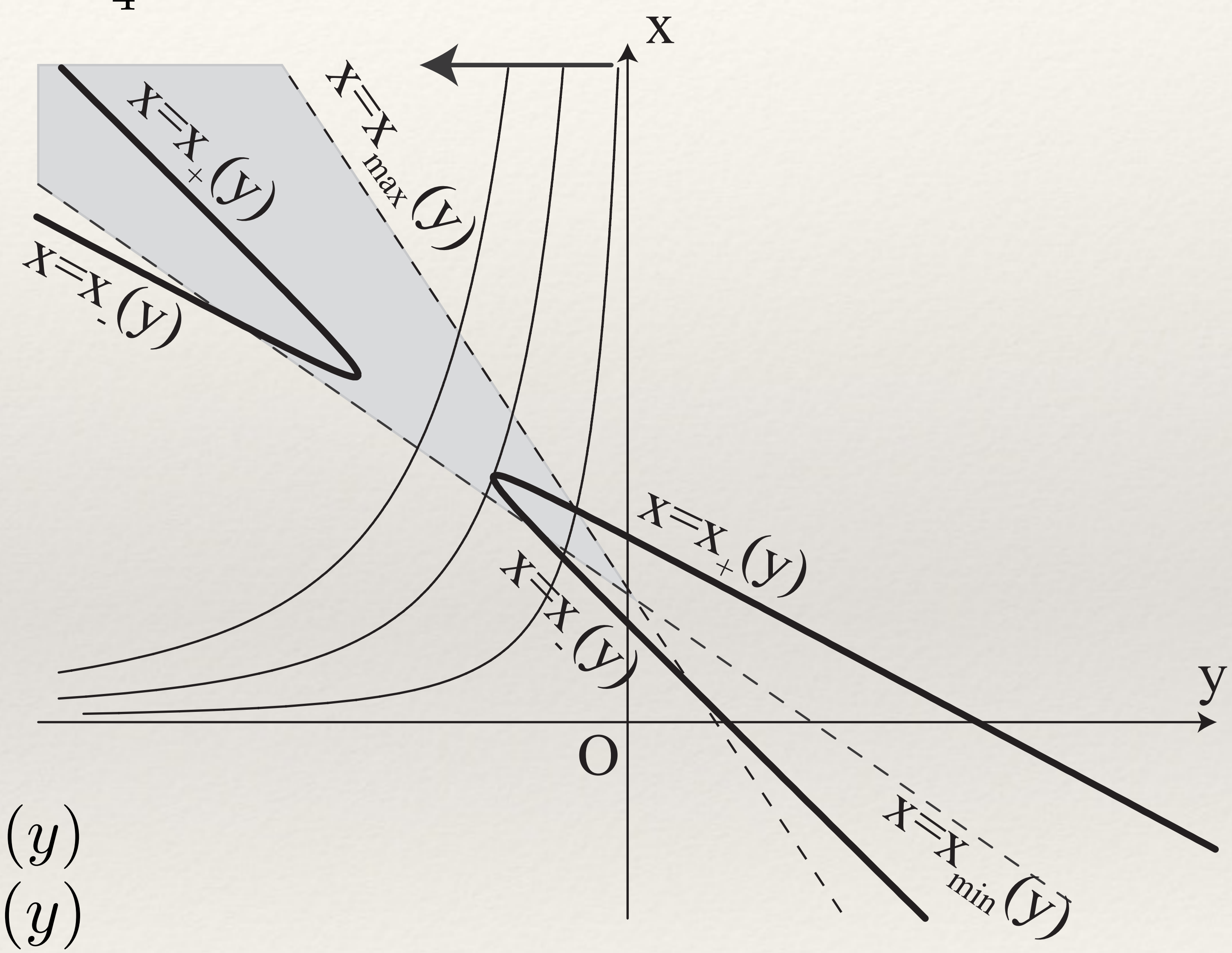
- We construct thin-shell wormholes made of its tension in the arbitrary dimensional spherically, planar, and hyperbolically symmetric spacetimes in both Einstein and EGB gravity.
- Spherical shell wormhole is unstable in both Einstein and EGB gravity.
- Small GB term destabilizes spherical wormhole.
- We gave analytic classification for all possible wormhole cases.

Appendix

Stability for $k = -1$ with $m < 0$

Static equation $(3 - 4\tilde{\alpha}\tilde{\Lambda})x^2 - 2(d-1)kxy + \frac{1}{4}(d-1)^2 y^2 + 16\tilde{\alpha}kx - 4d\tilde{\alpha}y + 16\tilde{\alpha}^2 = 0$

where $x := a_0^2, \quad y := \dfrac{m}{a_0^{d-5}} \; (< 0)$



$x = x_\pm(y)$ is solution to static eq.

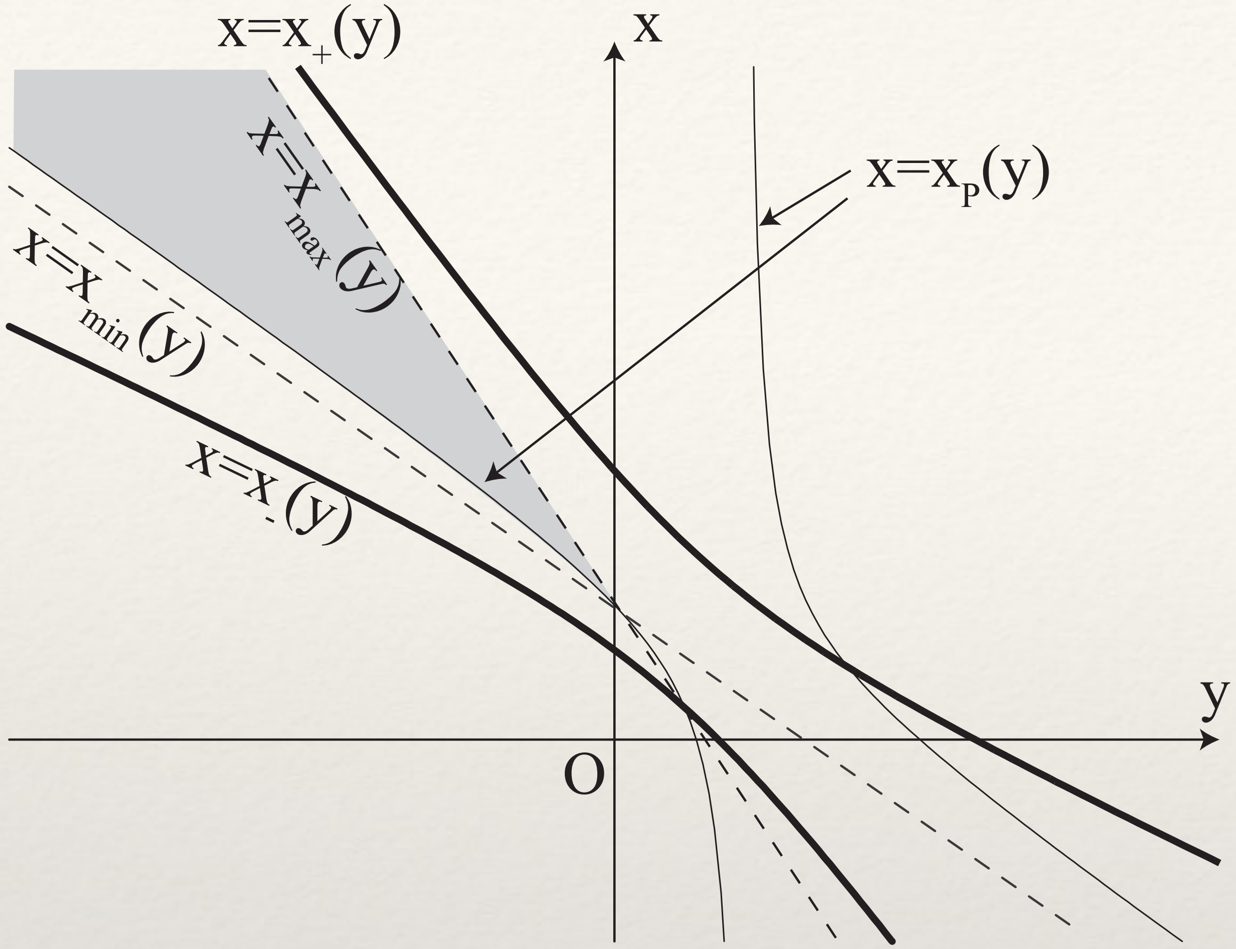
$x = x_{\mathrm{P}}(y)$ is solution to static eq. and represents marginal stability.

$f(a_0) > 0 \Rightarrow$ constraint for $x_{\max}(y)$

$\sqrt{A(a_0)} > 0 \Rightarrow$ constraint for $x_{\min}(y)$

$$f(a) = k + \frac{a^2}{2\tilde{\alpha}}\left(1 - \sqrt{A(a)}\right), \quad A(a) := 1 + 4\tilde{\alpha}\tilde{\Lambda} + \frac{4\tilde{\alpha}m}{a^{d-1}}.$$

$$\Lambda \geq 0$$



$x = x_{\max}(y)$ and $x = x_{+}(y)$ never intersect.

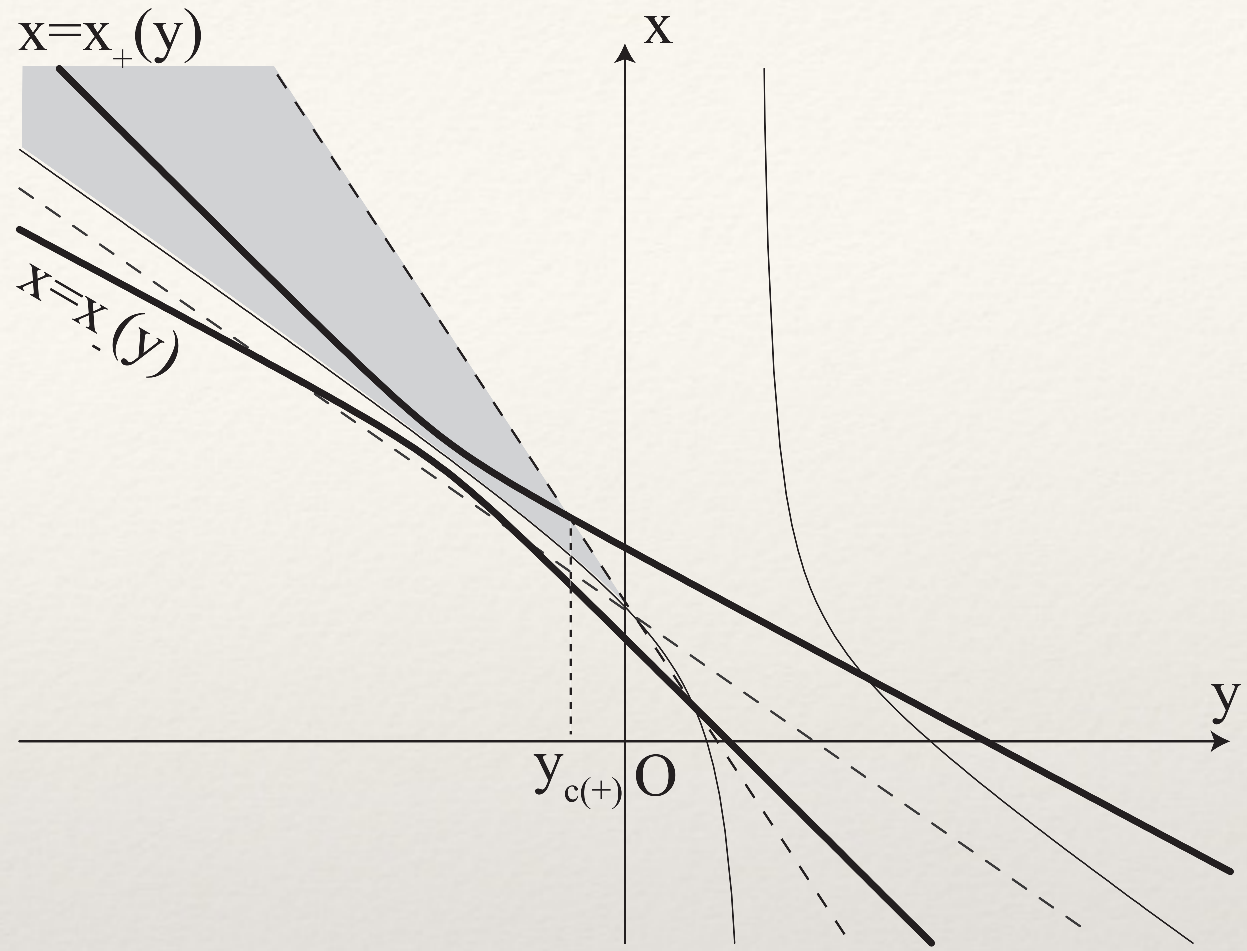
↓

$x = x_{+}(y)$ does not enter physical domain.

↓

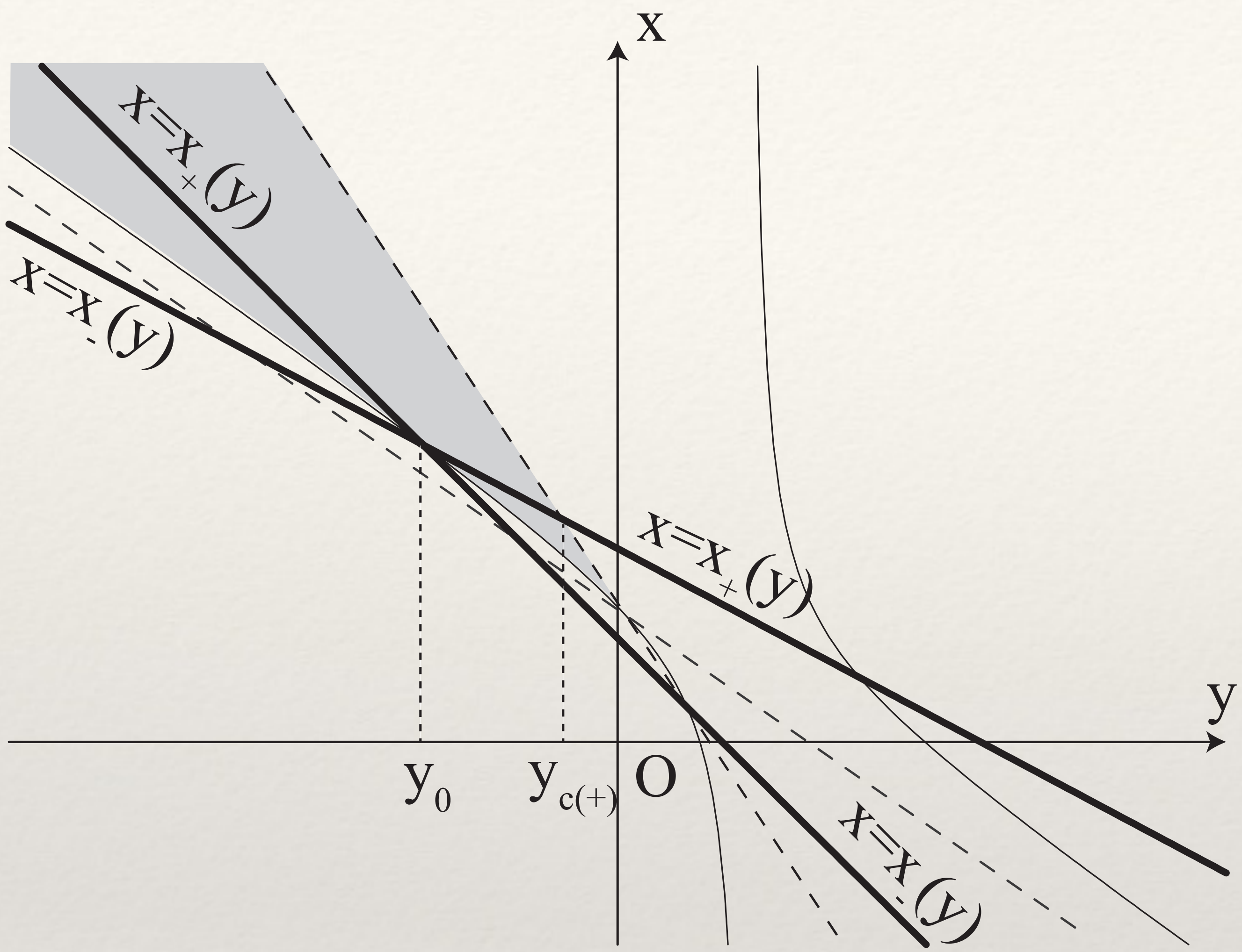
Solution does not exist

$$-\frac{2d-5}{2d-1} < 4\tilde{\alpha}\tilde{\Lambda} < 0$$



Any solution in the domain of $y < y_{c(+)}$ is stable.

$$\boxed{4\tilde{\alpha}\tilde{\Lambda} = -\frac{2d-5}{2d-1}}$$

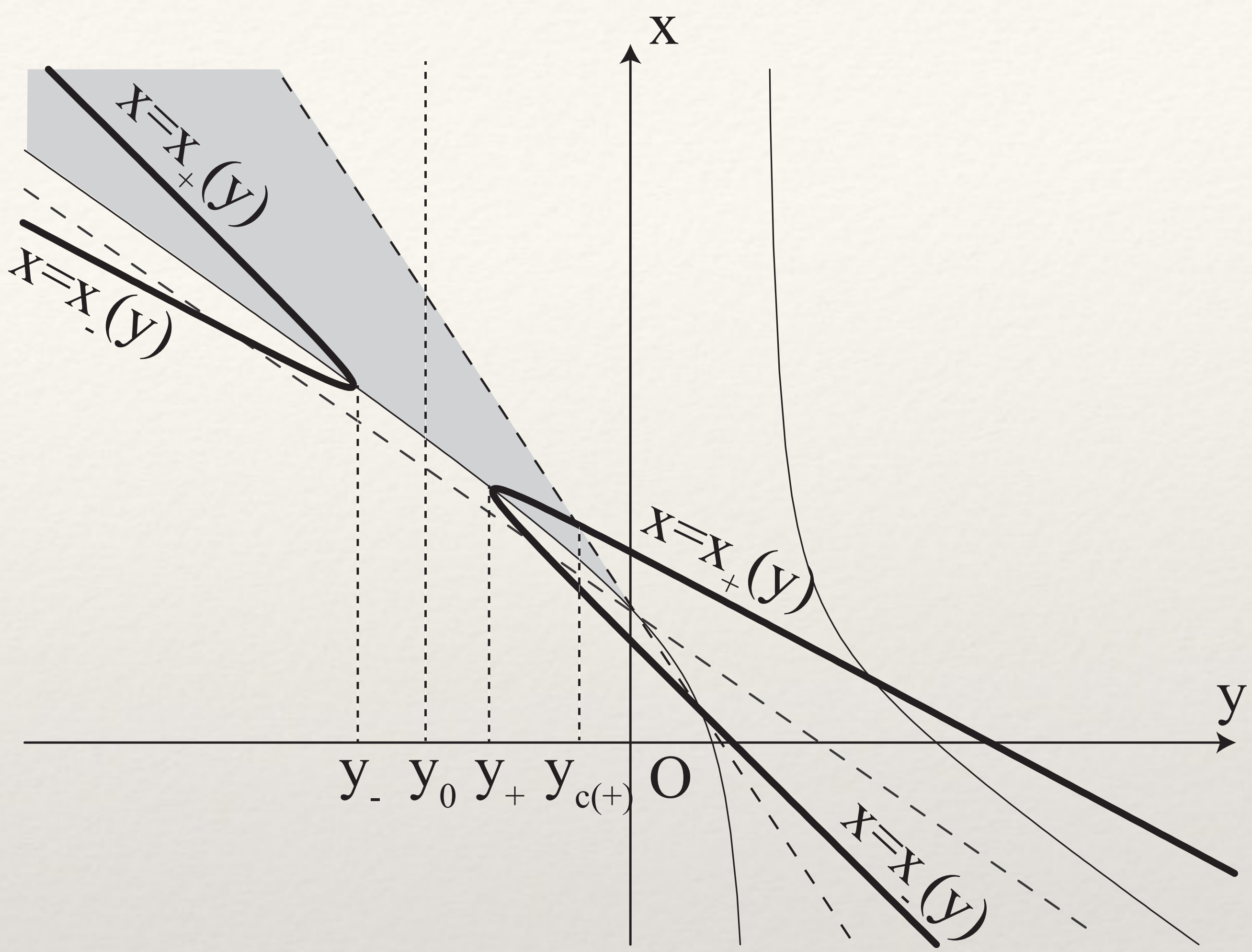


Intersection between $x = x_+(y)$ and $x = x_-(y)$ is located on $x = x_P(y)$

↓

Stable or marginally stable

$$\boxed{-1 < 4\tilde{\alpha}\tilde{\Lambda} < -\frac{2d-5}{2d-1}}$$



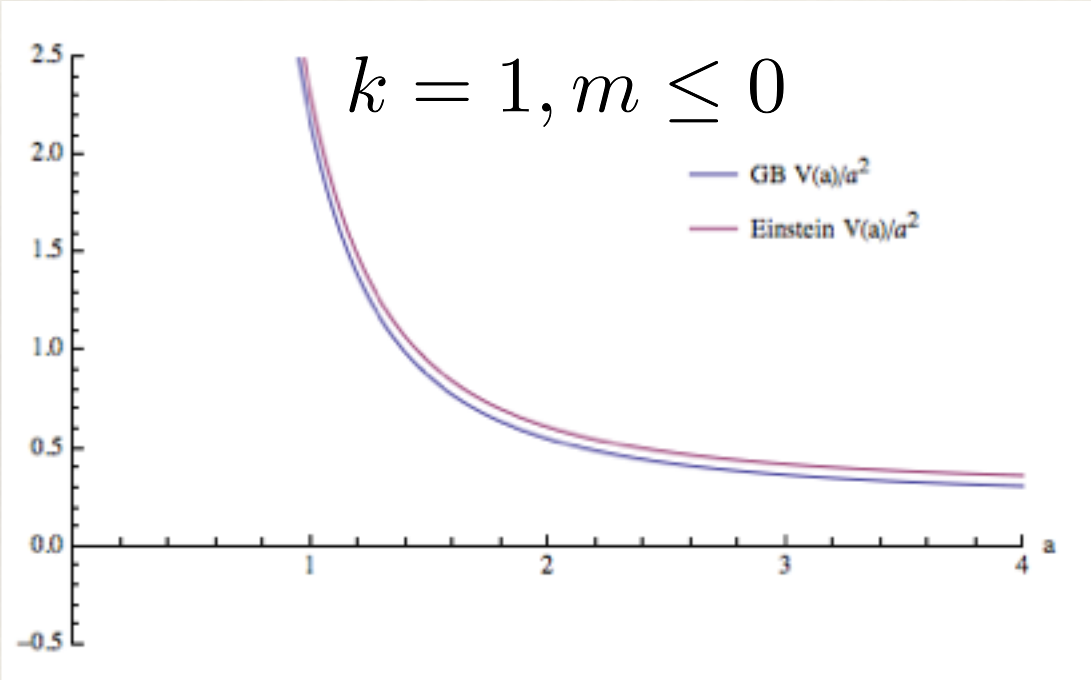
$d = 5$: Stable or marginally stable

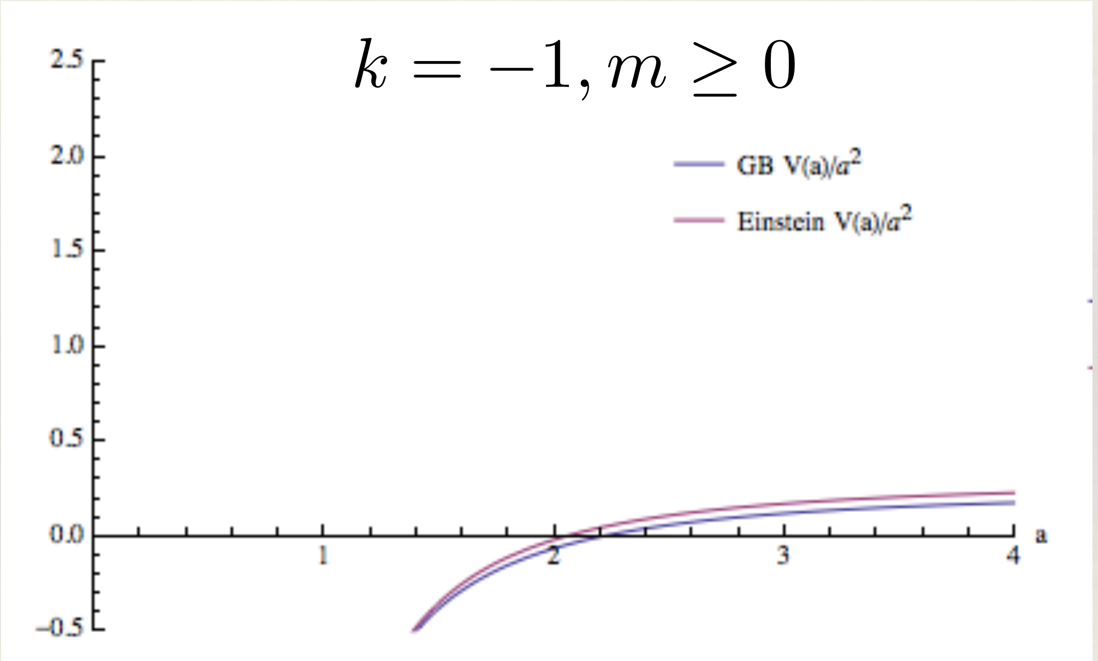
$d \geq 6$: Stable, marginally stable or unstable

Non-existence for $\begin{cases} k = 1 \text{ with } m \leq 0 \\ k = -1 \text{ with } m \geq 0 \end{cases}$

$$\bar{V}' = -\frac{1}{4\tilde{\alpha}B^2}\left\{\frac{8k\tilde{\alpha}B^2}{a^3} + (B^2 - A)B' + BA'\right\} \qquad \left(\bar{V} := \frac{V}{a^2}\right)$$

$$A'(a) = -\frac{4(d-1)\tilde{\alpha}m}{a^d}, \quad B'(a) = \frac{A(a)^{1/2}A'(a)}{2B^2}\left(1 + \frac{3\tilde{\alpha}\Omega^2}{\sqrt{\tilde{\alpha}\Omega^2(9\tilde{\alpha}\Omega^2 + A(a)^{3/2})}}\right)$$





$\bar{V}' < 0$ for $k = 1$ with $m \leq 0$.
$\bar{V}' > 0$ for $k = -1$ with $m \geq 0$.
} <u>no static wormhole</u>

Perfect fluid

For *k=1* with *m>0*, the wormhole with pure tension is unstable in both Einstein and EGB gravity.

→ how about more general matter?

→ *e.g.* perfect fluid

In Einstein gravity

$$V''(a_0) = -\frac{2(d-3)k}{a_0^2} - 2\frac{I(a_0)}{a_0^{d-2}}, \quad I(a_0) := (a^d \Omega \Omega')'_{a=a_0}, \quad \Omega(a)^2 := \frac{\kappa_d^4 \rho(a)^2}{4(d-2)^2}.$$

For *k=1*,

If $I(a_0) > 0$ is satisfied, wormhole is unstable.

Dust fluid (*p=0*) is such a matter field.

→ dust shell wormhole is unstable.

In Einstein-Gauss-Bonnet gravity

$$V''(a_0) = -\frac{P_3(a_0)}{a_0^2(3w_- + 4\tilde{\alpha}f_0)^3 w_+ w_-}, \quad w_\pm := a_0^2 + 2k\tilde{\alpha} \pm 2\tilde{\alpha}f_0 (>0),$$

$$P_3(a_0) := 2k(3w_- + 4\tilde{\alpha}f_0)^3 P(a_0) + 6a_0^7 P_2(a_0),$$

$$P_2(a_0) := (3w_- + 4\tilde{\alpha}f_0)^2 \left\{ w_- a_0^{-(d-1)} I(a_0) + 8\tilde{\alpha}k\Omega_0\Omega_0' \right\} + 12\tilde{\alpha}a_0^7(\Omega_0\Omega_0')^2,$$

$$P(a_0) := 4\tilde{\alpha}^2 f_0 \left\{ 6k - (d-3)f_0 \right\} + (a_0^2 + 2k\tilde{\alpha}) \left\{ (d-3)a_0^2 + 2(d-5)k\tilde{\alpha} \right\}$$

For *k=1*,

If $I(a_0) > 0$ is satisfied, wormhole is unstable:

$$\begin{aligned}
P_3(a_0) :=& 2(3w_- + 4\tilde{\alpha}f_0)^3 P(a_0) \\
&+ 6a_0^7 \left\{ (3w_- + 4\tilde{\alpha}f_0)^2 \left(w_- a_0^{-(d-1)} I(a_0) + 8\tilde{\alpha}\Omega_0\Omega_0' \right) + 12\tilde{\alpha}a_0^7(\Omega_0\Omega_0')^2 \right\} \\
>& 2(4\tilde{\alpha}f_0)^3 P(a_0) + 6a_0^7 \left\{ (4\tilde{\alpha}f_0)^2 (8\tilde{\alpha}\Omega_0\Omega_0') + 12\tilde{\alpha}a_0^7(\Omega_0\Omega_0')^2 \right\} \\
>& 2(4\tilde{\alpha}f_0)^3 16\tilde{\alpha}^2 f_0 + 6a_0^7 \left\{ (4\tilde{\alpha}f_0)^2 (8\tilde{\alpha}\Omega_0\Omega_0') + 12\tilde{\alpha}a_0^7(\Omega_0\Omega_0')^2 \right\} \\
=& 8\tilde{\alpha} \left\{ (4\tilde{\alpha}f_0)^2 + 3a_0^7\Omega_0\Omega_0' \right\}^2 > 0
\end{aligned}$$

Dust fluid (*p=0*) is such a matter field.

$\rightarrow$ dust shell wormhole is unstable.