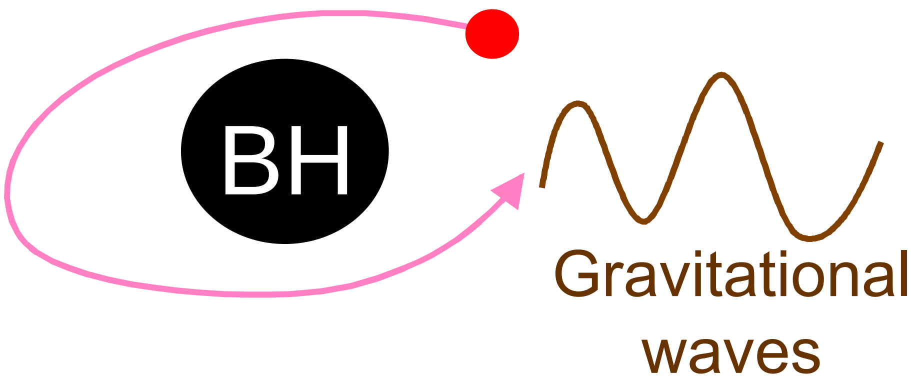
Adiabatic approach to the second order orbital evolution in black hole spacetime



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partly based on collaboration with
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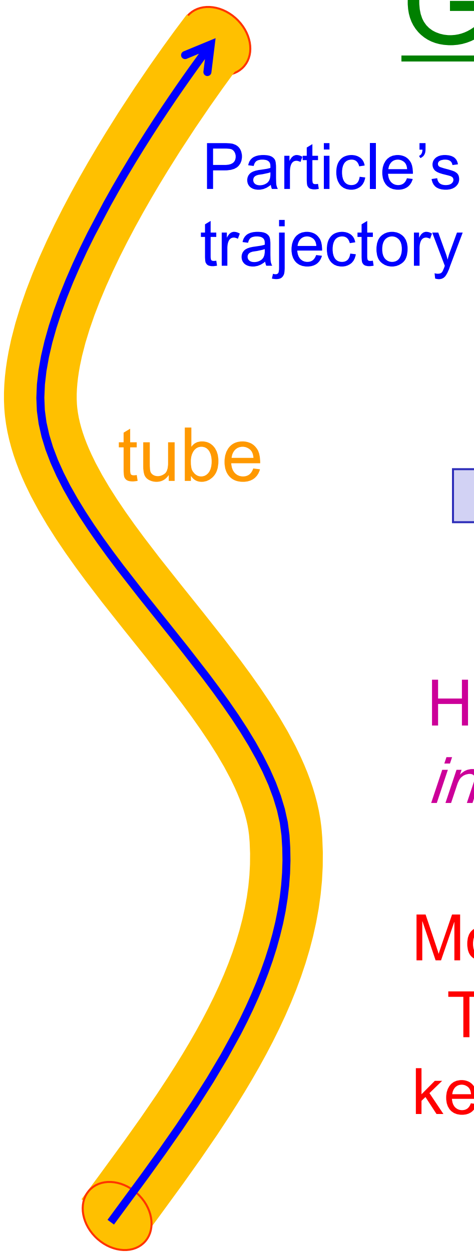
Gauge invariance



Perturbation should be small everywhere outside the world tube

"tube radius" >> μ (mass of satellite)

⇒ Unavoidable ambiguity in the perturbed trajectory of $O(\mu)$

However, "long term orbital evolution is *gauge invariant*", up to the above ambiguity of $O(\mu)$.

Motivation:
There must be a concise description keeping only the gauge invariant information

Use of canonical transformation

need to solve the geodesic equation on perturbed regularized spacetime

$$S = \frac{1}{2}\int g^{\mu\nu} u_\mu u_\nu d\tau = \frac{1}{2}\int g_{(0)}^{\mu\nu} u_\mu u_\nu d\tau \underbrace{- \frac{1}{2}\int h_{full}^{\mu\nu} u_\mu u_\nu d\tau}_{\text{interaction Hamiltonian } H_{\mathrm{int}}}$$

Change the variables to the "action variables" J_a
(~ constants of motion in the background $\{u^2/2, -E, L_z, Q\}$)
+ their conjugate "angle variables" w^a ,
using the standard canonical transformation.

Generating function:

$$W(x, J) = J_t t + J_\phi \phi + \int^r \underline{\tilde{u}_r(r', J)}\, dr' + \int^\theta \underline{\tilde{u}_\theta(\theta', J)}\, d\theta'$$

well-known fns for Kerr geodesic motion

$$J_r = \oint \tilde{u}_r(r', J) dr' \qquad J_\theta = \oint \tilde{u}_\theta(\theta', J) d\theta'$$

Radiation reaction to the action variables (constants of motion)

"retarded" field= "radiative" + "symmetric"

$$\frac{\text{"ret"} - \text{"adv"}}{2} \qquad \frac{\text{"ret"} + \text{"adv"}}{2}$$

regularization
is unnecessary

$$\left\langle \frac{dJ_\alpha}{d\tau} \right\rangle = \left\langle \frac{\partial H_{\rm int}}{\partial w^\alpha} \right\rangle \approx \int d\tau \int d\tau' \frac{\partial}{\partial w^\alpha} G^{(ret)}(\gamma, \gamma') \Big|_{\gamma'=\gamma}$$

1) $\partial / \partial w^a$ can be replaced with $\partial / \partial w^a_{(ini)}$.

initial value

2) $\int d\tau \int d\tau' G^{(ret)}(\gamma, \gamma') \Big|_{\gamma'=\gamma}$ after substitution $\gamma' = \gamma$
is independent of $w^a_{(ini)}$.

At the leading order in $\eta = \mu/M$, only the radiative part determines the change of "constants of motion",
except for resonance orbits.

(Mino (2003))

Gauge invariance of the angular velocity

Angle variables $w^a = O(\eta^{-1})$ ($\propto$ radiation reaction time) are gauge invariant in the context of long term evolution.

allowing $O(\eta)$ gauge ambiguity, which does not accumulate.

$\Rightarrow$ $\omega^a = \langle \dot{w}^a \rangle = O(\eta^0)$ should be invariant up to $O(\eta)$

$\because$

$$W(x,J) = J_t t + J_\phi \phi + \int^r \tilde{u}_r(r',J)dr' + \int^\theta \tilde{u}_\theta(\theta',J)d\theta'$$

$$J_r = \oint \tilde{u}_r(r',J)dr' \qquad J_\theta = \oint \tilde{u}_\theta(\theta',J)d\theta'$$

$$W(x,J) = \tilde{W}(x,J) + n_r J_r + n_\theta J_\theta$$

where we introduce a single valued function with respect to x:

$$\tilde{W}(x,J) = J_t t + J_\phi \phi + \int_{r_0}^r \tilde{u}_r(r',J)dr' + \int_{\theta_0}^\theta \tilde{u}_\theta(\theta',J)d\theta'$$

$$w^I = \frac{\partial W(x,J)}{\partial J_I} = \frac{\partial \tilde{W}(x,J)}{\partial J_I} + n_I \quad (I = r,\theta) \qquad w^i = \frac{\partial W(x,J)}{\partial J_i} = \frac{\partial \tilde{W}(x,J)}{\partial J_i} \quad (i = t,\phi)$$

Small variations of x and J are not amplified in w.

Long term evolution

EOM)

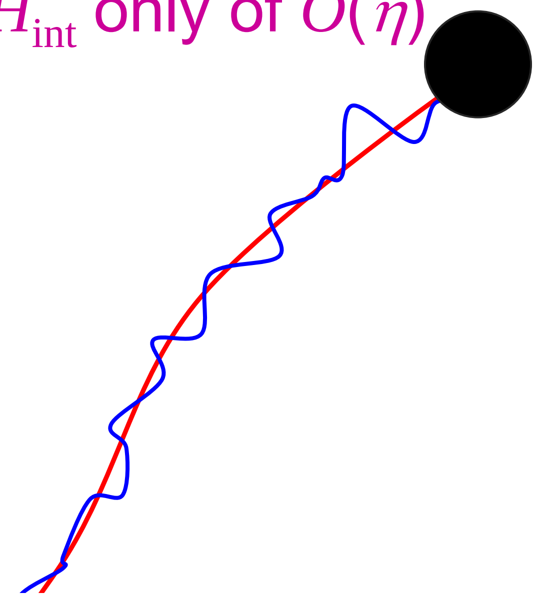
We wish to avoid errors in w^a of $O(\eta^0)$

$$\frac{dJ_a}{d\tau} = -\frac{\partial H_{\mathrm{int}}}{\partial w^a}$$

We need H_{int} of $O(\eta)$+$O(\eta^2)$

$$\frac{dw^a}{d\tau} = \frac{\partial H^{(0)}}{\partial J^a} + \frac{\partial H_{\mathrm{int}}}{\partial J^a}$$

We need H_{int} only of $O(\eta)$



We decompose the variables

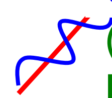
$$J_a = \bar{J}_a + \delta J_a$$
$$w^a = \bar{w}^a + \delta w^a$$

so that

$(\bar{J}, \bar{w})$:slowly changing trend

$(\delta J, \delta w)$:rapidly oscillating but always small

can be eliminated by gauge transformation



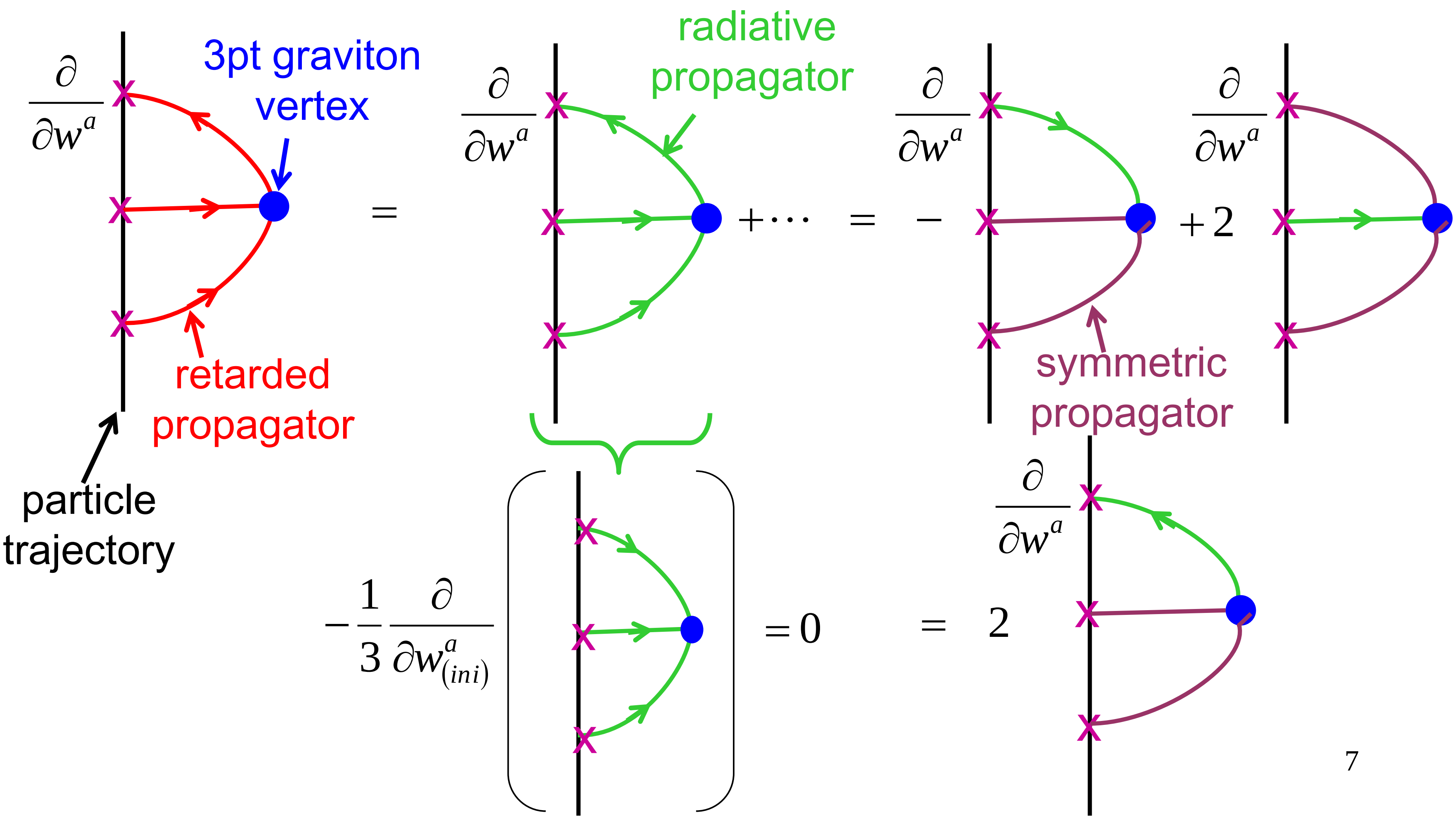
(arXiv:160x.xxxxx, Fujita, Isoyama, Le Tiec, Nakano, Sago, TT)

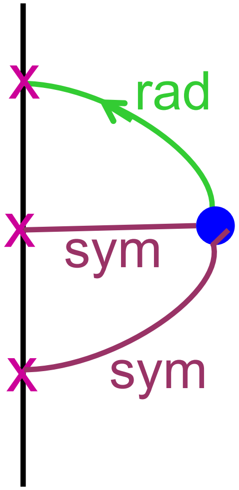
Drastic simplification

From here on I give arguments that are not approved by all collaborators.

$$\frac{dJ_a}{d\tau} = -\frac{\partial H_{\text{int}}}{\partial w^a} = -\frac{\partial H_{\text{int}}^{(1)}}{\partial w^a} - \frac{\partial H_{\text{int}}^{(2)}}{\partial w^a}$$

Second order dissipative part = Graviton 2-loop



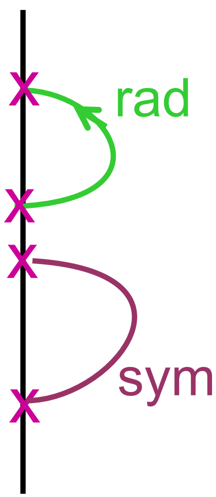
$\frac{\partial}{\partial w^a}$ 

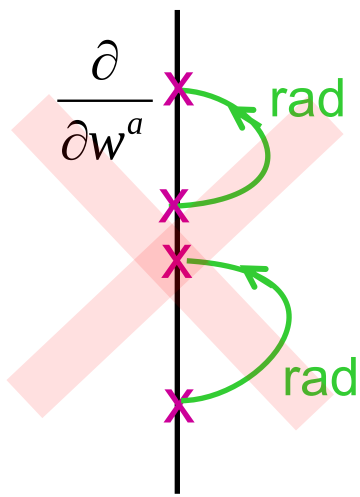
$$= \int d\tau \int d\tau' \int d\tau'' \int d^4 x' \times \hat{V}_{x'} \frac{\partial}{\partial w^a} G_{(rad)}\left(x, x_1'\right)\Big|_{x=\gamma(\tau)} G_{(sym)}\left(x_2', \gamma(\tau')\right) G_{(sym)}\left(x_3', \gamma(\tau'')\right)$$

Second order source must satisfy conservation $T_{\mu\nu}{}^{;\nu}=0$ as a whole

$$\Rightarrow \quad \underline{\nabla^{\nu} T^{(\text{particle})}_{\mu\nu}} = \nabla^{\nu} G^{[2]}_{\mu\nu}\left(h_{(sym)}, h_{(sym)}\right)$$

necessary deviation from geodesic comes from symmetric contribution only.

$\Rightarrow$ $\frac{\partial}{\partial w^a}$ 

$\frac{\partial}{\partial w^a}$ 

$$= \int d\tau \frac{\partial}{\partial w^a} \int d\tau' \delta J_b^{(sym)} \frac{\partial}{\partial J_b} G_{(rad)}\left(x, \gamma(\tau')\right)$$

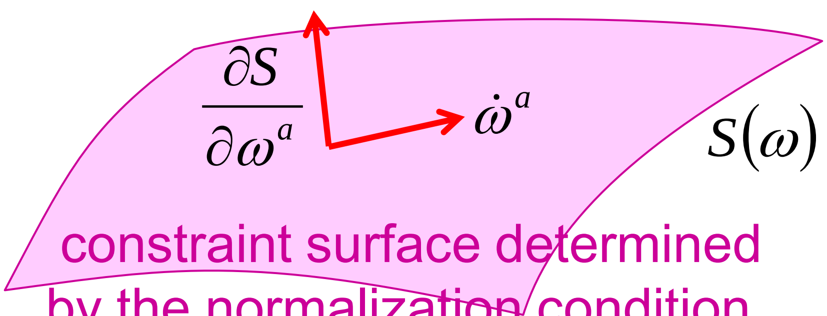
The right hand side of $\omega^a = \dfrac{\partial H^{(0)}}{\partial J^a} + \dfrac{\partial H_{\text{int}}}{\partial J^a}$

as a function of J_a is gauge dependent

However, $\dot{J}_a$ evaluated as mentioned above is gauge invariant

⇨ Special choice $J_a(\omega)$ should exist!

$\omega^a J_a(\omega) = g^{\mu\nu} u_\mu u_\nu = -1$: normalization condition--★



$S(\omega) = \text{constant}$

$\omega^a \dot{J}_a = -\left\langle \dfrac{dw^a}{d\tau} \dfrac{\partial H}{\partial w^a} \right\rangle = 0$ ⇨ $\dot{\omega}^a J_a = 0$ for any $\dot{\omega}^a$ tangential to the constraint surface

⇨ $J_a \propto \dfrac{\partial S}{\partial \omega^a}$, which uniquely specifies $J_a(\omega)$ with ★.

Conclusion

We discussed a method for long-term evolution that controls the error of the orbital phase to be $\mathrm{O}(\eta = \mu/M)$.

$$\frac{dJ_a}{d\tau} = -\frac{\partial H_{\mathrm{int}}}{\partial w^a}$$

second order perturbation can be calculated as

$$h^{\mu\nu}_{(2)} = \int d\tau \int d^4x' G^{\mu\nu\rho\sigma}_{(rad)}(x, x') T^{(eff)}_{\rho\sigma}\left[h_{(sym)}, \gamma + \delta\gamma_{(sym)}\right](x')$$

NOTICE: Field determined by radiative propagator is independent of how we remove the singular part of the source as long as it is written schematically in the form of "$\Box h$".

$$\omega^a = \frac{\partial H^{(0)}}{\partial J^a} + \frac{\partial H_{\mathrm{int}}}{\partial J^a}$$

is necessary only to determine the constraint surface specified by $\omega^a j_a(\omega) = -1$

It can be also shown that the constraint surface is determined by the time-average of H_{int}, which is gauge invariant.

There is also an issue about the boundary condition near the horizon and infinity, but the goal might be close.