Compact objects in Lovelock gravity theory

Sudan Hansraj

Astrophysics and Cosmology Research Unit,
School of Mathematical Sciences,
University of KwaZulu-Natal,
Durban

21st International Conference on General Relativity and Gravitation
July 11 - 15, 2016 New York City, USA

Motivation for Exact Solutions

♠ Models of compact stars

♠ compact stars as nuclear physics labs - EOS determination from Mass–Radius relationships

♠ compact stars tell us about strong gravity regimes - GR - not suitable; rule out other theories

Einstein–Hilbert Action

$$S = \frac{1}{2\kappa}\int R\sqrt{-g}\,\mathrm{d}^4 x$$

where

$$g = \det\left(g_{\mu\nu}\right)$$

$$\kappa = 8\pi G c^{-4}$$

$$R = \text{Ricci scalar}$$

Lovelock Action

- $$\text{Lagrangian} \qquad \mathcal{L} = \sum_{n=0}^{t} \alpha_n \mathcal{R}^n$$

- $$\mathcal{R}^n = \frac{1}{2^n} \delta^{\mu_1 \nu_1 \dots \mu_n \nu_n}_{\alpha_1 \beta_1 \dots \alpha_n \beta_n} \Pi_{r=1}^{n} R^{\alpha_r \beta_r}_{\mu_r \nu_r}$$

- $$R^{\alpha\beta}_{\mu\nu} = \text{Riemann tensor}$$

- $$\delta^{\mu_1 \nu_1 \dots \mu_n \nu_n}_{\alpha_1 \beta_1 \dots \alpha_n \beta_n} = \frac{1}{n!} \delta^{\mu_1}_{[\alpha_1} \delta^{\nu_1}_{\beta_1} \dots \delta^{\mu_n}_{\alpha_n} \delta^{\nu_n}_{\beta_n]}$$

Special Cases

$N = 0$ - cosmological constant Λ
$N = 1$ - Einstein
$N = 2$ - Einstein–Gauss–Bonnet

$$\mathcal{L} = \sqrt{-g}\left(\alpha_0 + \alpha_1 R + \alpha_2\left(R^2 + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu}\right) + \alpha_3\mathcal{O}(R^3)\right)$$

Quadratic Gauss-Bonnet Term $\quad \mathcal{R}^2 = R^2 + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu}$

quadratic term is present in the low energy effective action of heterotic string theory

Motivation for pure Lovelock gravity

pure Lovelock = only *Nth* order Lovelock term in Lagrangian

large scales - modify GR - explain accelerated expansion of universe

Special case - GB term appears in low energy limit of superstring theory

Gives second order quasilinear equations of motion (no ghost)

First order is GR - contains the gains of GR

Lovelock Field Equation

$$\sum_{N=0}^{N} \alpha_N \mathcal{G}_{AB}^{(N)} = \sum_{N=0}^{N} \alpha_N \left(N \left(\mathcal{R}_{AB}^{(N)} - \frac{1}{2} \mathcal{R}^{(N)} g_{AB} \right) \right) = T_{AB} \tag{1}$$

where $\mathcal{R}_{AB}^{(N)} = g^{CD} \mathcal{R}_{ACBD}^{(N)}, \mathcal{R}^{(N)} = g^{AB} \mathcal{R}_{AB}^{(N)}$, and T_{AB} is the energy momentum tensor.

Static d dimensional line element

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2_{d-2} \tag{2}$$

$d\Omega^2_{n-2}$ is the metric on a unit $(d-2)$-sphere and

$\nu = \nu(r)$ and $\lambda = \lambda(r)$ are the metric potentials.

Energy momentum tensor

comoving fluid velocity

$u^a = e^{-\nu/2}\delta^a_0$

$T^a_b = \mathrm{diag}\left(-\rho,\, p_r,\, p_\theta,\, p_\phi,\, ...\right)$

Lovelock Field Equations

$$\rho = \frac{(d-2)e^{-\lambda}\left(1-e^{-\lambda}\right)^{N-1}\left(rN\lambda' + (d-2N-1)(e^{\lambda}-1)\right)}{2r^{2N}} \tag{3}$$

$$p_r = \frac{(d-2)e^{-\lambda}\left(1-e^{-\lambda}\right)^{N-1}\left(rN\nu' - (d-2N-1)(e^{\lambda}-1)\right)}{2r^{2N}} \tag{4}$$

$$\begin{aligned} p_\theta = {} & \frac{1}{4}r^{-2N}\left(e^{\lambda}\right)^{-N}\left(e^{\lambda}-1\right)^{N-2}\left[-Nr\lambda'\left\{2(d-2N-1)\left(e^{\lambda}-1\right)\right.\right. \\ & \left.+r\nu'\left(e^{\lambda}-2N+1\right)\right\} + \left(e^{\lambda}-1\right)\left\{-2(d-2N-1)(d-2N-2)\left(e^{\lambda}-1\right)\right. \\ & \left.\left.+2N(d-2N-1)r\nu' + Nr^2\nu'^2 + 2Nr^2\nu''\right\}\right] \end{aligned} \tag{5}$$

Conservation Laws

$T^{ab}_{;b} = 0$

$$\frac{1}{2}\left(p_r + \rho\right)\nu' + p_r' + \frac{(d-2)}{r}\left(p_r - p_\theta\right) = 0 \tag{6}$$

Pressure Isotropy

$$\begin{aligned} r\lambda' \left\{2(d-2N-1)\left(e^{\lambda}-1\right)+r\nu'\left(e^{\lambda}-2N+1\right)\right\} & \\ -\left(e^{\lambda}-1\right)\left\{4(d-2N-1)\left(e^{\lambda}-1\right)+r\left(\nu'\left(-4N+r\nu'+2\right)+2r\nu''\right)\right\} &= 0 \end{aligned} \tag{7}$$

Transformed field equations

use transformations: $e^{\nu(r)} = y^2(x)$ and $e^{-\lambda(r)} = Z(x)$ where $x = Cr^2$, C a constant.

$$
\begin{aligned}
\rho &= \frac{C^N(d-2)(1-Z)^{N-1}\left[(d-2N-1)(1-Z) - 2Nx\dot{Z}\right]}{2x^N} && (8)\\
p &= \frac{C^N(d-2)(1-Z)^{N-1}\left[4NxZ\dot{y} - (d-2N-1)(1-Z)y\right]}{2x^N y} && (9)\\
0 &= 4x^2 Z(1-Z)\ddot{y} + \left[4(1-N)xZ(1-Z) + 2x^2\left(1-(2N-1)Z\right)\dot{Z}\right]\dot{y} \\
&\quad + (d-2N-1)(1-Z)\left(\dot{Z}x - Z + 1\right)y && (10)
\end{aligned}
$$

Elementary Physical Conditions

♠ Continuity of fundamental forms across a pressure-free hypersurface

not completely understood - we use conditions extrapolated from Einstein: $p(R) = 0$ (cf Davis for brane world junction conditions).

♠ For EGB match g_{00} and g_{11} with exterior Boulware–Deser (1985) metric

$$e^{2\nu(R)} = 1 + \frac{R^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8M\alpha}{R^4}}\right) = e^{-2\lambda(R)}$$

Lovelock exterior solution (Whitt/ Wheeler) $ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{d-2}$ where $f(r) = 1 - \left(\frac{M}{r}\right)^{1/N}$ for $d = 2N + 2$.
(for $d = 2N + 1$ gravity not dynamic - no non-trivial vacuum solution)

♠ causality criterion: $0 < \frac{dp}{d\rho} < 1$

continued

♠ positive definite and finite energy and pressure - everywhere within radius
- monotonic decrease of pressure and density from centre to boundary

♠ stability with respect to radial perturbations

♠ Compliance with the energy conditions:

- weak energy condition: $\rho - p > 0$
- strong energy condition: $\rho + p > 0$
- dominant energy condition: $\rho + 3p > 0$

Recent New Results for Compact objects in EGB

explicit exact interior metrics:

$y = a + bx^2$

Hansraj, Maharaj and Chilambwe: 2015 *Eur. Phys. J. C (2015) 75: 277*

Z a constant

Maharaj, Chilambwe and Hansraj 2015 *Phys. Rev. D (2015) 91, 084049*

$y = 1 + x$
Chilambwe, Hansraj and Maharaj *Int Journal of Mod Physics* **24**, *07, (2015)*

Isothermal Fluid Sphere $\rho \sim \frac{1}{r^2}$; $p = \alpha\rho$

$\lambda =$ constant gives $\rho \sim 1/r^{2N}$.

CONVERSE? also true.

constant λ is necessary and sufficient for isothermal

The case $d = 2N + 1$

$$e^{-\lambda} = 1 - (k_1 \ln r + k_2)^{1/N}. \tag{11}$$

pressure isotropy only satisfied for $k_1 = 0$.

So λ constant gives $\rho = 0$.

no isothermal sphere in the critical $d = 2N + 1$ dimension.

The general case $Z =$ constant

isotropy equation (7) which for $e^{\lambda} = k$ reduces to

$$2r^2\nu'' + r^2\nu'^2 - 2(2N-1)r\nu' + 4(d-2N-1)(k-1) = 0. \qquad (12)$$

Riccati type equation - general solution

$$e^{\nu} = c_2 r^{2\left(N-\sqrt{A}\right)}\left(c_1 + r^{2\sqrt{A}}\right)^2 \qquad (13)$$

where $A = (d-2N-1)(1-k) + N^2$ and c_1 and c_2 are constants of integration.

$$p = \frac{(d-2)(k-1)^{N-1}\left(c_1\left(N-\sqrt{A}\right)^2 + r^{2\sqrt{A}}\left(N+\sqrt{A}\right)^2\right)}{2\left(r^{2\sqrt{A}} + c_1\right)k^N r^{2N}} \qquad (14)$$

Can be shown to only admit the isothermal case. (Dadhich, Hansraj and Maharaj (2016) Phys. Rev. D 93, 044072)

Incompressible fluid sphere

in Einstein gravity - constant density = Schwarzschild (1916) solution.

In pure Lovelock remarkably solution is still Schwarzschild.

Corresponds to the choice $Z = 1 + x$

pressure isotropy independent of spacetime dimension d and Lovelock polynomial order N.

Finch–Skea Potential

$Z = \frac{1}{1+x}$

isotropy equation:

$$4(1+x)\ddot{y} - 2(2N-1)\dot{y} + (d-2N-1)y = 0. \tag{15}$$

general exact solution

$$\begin{aligned} y &= (1+x)^{(2N+1)/4}\left(c_1 J_{-N-\frac{1}{2}}\left(\sqrt{(d-2N-1)(1+x)}\right)\right. \\ &\quad \left. + c_2 Y_{-N-\frac{1}{2}}\left(\sqrt{(d-2N-1)(1+x)}\right)\right) \end{aligned} \tag{16}$$

J and Y are Bessel functions - half-integer order - elementary functions

Special case: 5-d pure Gauss–Bonnet

$e^{-\lambda} = \frac{1}{1+Cr^2}$

$$e^{\nu} = c_1(1 + Cr^2)^{\frac{5}{2}} + c_2 \tag{17}$$

energy density

$$\rho = \frac{12C^3}{(1 + Cr^2)^3} \tag{18}$$

pressure

$$p = \frac{60C^3}{(1 + Cr^2)^{\frac{1}{2}}\left((1 + Cr^2)^{\frac{5}{2}} + \kappa\right)} \tag{19}$$

$\kappa = \frac{C_2}{C_1}$.

GB case in 6-d

$$e^{\lambda} = v \tag{20}$$

$$e^{\nu} = \left(a(3 - v^2) - 3bv\right)\sin v + \left(b(v^2 - 3) - 3av\right)\cos v \tag{21}$$

$$\rho = \frac{12(v^2 + 4)}{v^6} \tag{22}$$

$$p = \frac{2\left[(\zeta v - v^2 - 1)\tan v + \zeta(v^2 + 1) + v\right]}{v^6\left[(v^2 - 3 + 3\zeta v)\tan v - \zeta\,(v^2 - 3) + 3v\right]} \tag{23}$$

$\zeta = \frac{a}{b}$ and put $C = 1$.

Summary and Conclusion

- derived explicit pure Lovelock field equations
- isothermal fluid sphere universal in all d and N
- Schwarzschild interior metric holds - constant density - isotropy equation has no d and N.
- Lovelock 'spheres' of order $d = 2N + 1$ - unbounded
- Compact fluid spheres exist for $d = 2N + 2$
- Constructed exact models for $N = 2$ ($d = 6$) Gauss–Bonnet gravity

References

- D Lovelock *J. Math. Phys.* **12** 498 (1971).
- D Boulware, S Deser, Phys. Rev. Lett. **55**, 2656 (1985).
- B Whitt, Phys. Rev. D **38**, 3000 (1988).
- J T Wheeler, Nucl. Phys. B **268**, 737 (1986); **27**, 732 (1986)
- R C Myers and J Simon, Phys. Rev. D **38**, 2434 (1988)
- D L Wiltshire, Phys. Rev. D 38, 2445 (1988)
- N Dadhich, A Molina, A Khugaev, Phys. Rev. **81**, 104026 (2010); arxiv:1001.3952
- S D Maharaj, B. Chilambwe and S Hansraj, Phys. Rev. D **91**, 084049 (2015)
- S Hansraj, B Chilambwe and S D Maharaj, Eur. Phys. J. C **27** 277 (2015)
- B Chilambwe, S. Hansraj and S D Maharaj, Int. J. Mod. Phys. D **24** 1550051 (2015)
- N. Dadhich, S Hansraj, S D Maharaj, Phys. Rev. D **93** *to appear*
- M R Finch and J E F Skea *Class. Quant. Grav.* **6** 467 (1989)
- S C Davis, Phys. Rev. D **67**, 024030 (2002)