The structure of spaces of null geodesics

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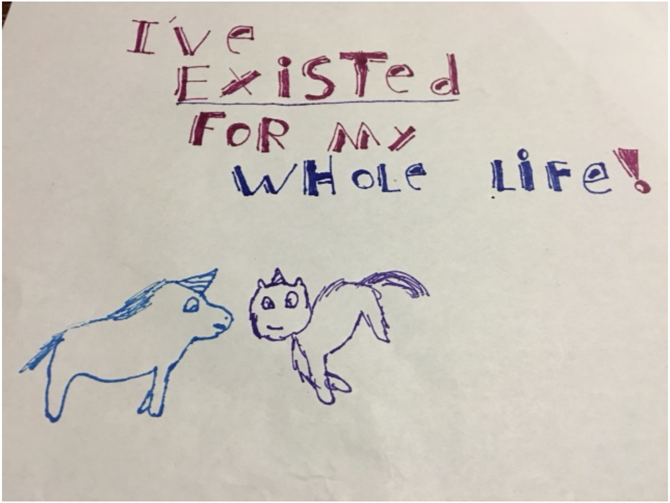
Do unicorns exist?



Don Quixote by Gustav Doré

Penrose Tribars at a defunct school in Seattle

An impossible coincidence.

Looking for complex structure

Relativists have been very successful in studying certain kinds of space-times, where because of some structural feature of the space-time in question, *complex numbers and complex analysis* emerge as critical to the study of the geometry.

- This is very apparent, for example in the study of algebraically special space-times, where at the least *Cauchy-Riemann structures* (those structures that the boundary of a complex manifold inherits from the ambient complex structure) play a big role.
- In the case of *self-dual space-times* (which are necessarily complex a priori, or real with non-Lorentzian metrics), applications of complex analysis have led to spectacular progress, for example in the construction of instantons.

There are enough examples of this kind that we are tempted to ask if there are aspects of complex analysis present in essentially any space-time.

From null geodesics to complex numbers

In the present talk, we will pinpoint three features of the structure of null geodesics, that appear to reveal a role for complex numbers. One is present in any dimension, the other two, at least in their present formulation, work only in dimension four.

- We show that any null geodesic in four dimensional space-time has a natural complexification.
- Using the theory of *Penrose* limits, we assign to any null geodesic in four-dimensional space-time a curve in the complex (upper-half) plane. This opens up the possibility of using uniformization theorems based on conformal transformation theory, to tackle space-time geometry.
- Finally, we will see, in any dimension, that the *moduli space* of Penrose limits is at least as complicated as Universal Teichmüller space, so at least as complicated as string theory.

What are null geodesics?

Null geodesics model the limits of causality in space-time.
A space-time point that is connected by a null geodesic to a point in the future can send a signal to that point, but *only just*: the signal has to travel at the speed of light (also the speed of gravity). Mathematically they are easy to define.

- Following Einstein, space-time $\mathcal{M}$ is equipped with a smooth Lorentzian metric g of signature $(1, n-1)$ in n-dimensions.
- On $\mathcal{T}^*\mathcal{M}$, the co-tangent bundle of $\mathcal{M}$, the metric gives a Hamiltonian function, $H = 2^{-1}g^{-1}(p, p)$ at each point (x, p), with $x \in \mathcal{M}$ and p a co-vector at x.
- Using the natural Poisson structure $\{,\}$ of $\mathcal{T}^*\mathcal{M}$, the Hamiltonian gives rise to a vector field H', such that $H'(f) = \{H, f\}$, for any smooth function f on $\mathcal{T}^*\mathcal{M}$.
- H is constant along the flow of H'; the null geodesics are just the curves of the flow along which H vanishes.

Nuances

The curves given by the Hamiltonian flow are naturally parametrized, such that their tangent vectors are parallelly propagated along the curves. It is often convenient to ignore this parametrization, which can be achieved for example by identifying trajectories through the same family of space-time points, without selecting a particular tangent vector.

- A key fact is that the trajectories are conformally invariant.

If g is replaced by fg, with f smooth and non-zero, then $H \to f^{-1}H$ and $\{f^{-1}H, \} = f^{-1}\{H, \} - Hf^{-2}\{f, \}$, so we get the relation $(f^{-1}H)' = f^{-1}H' - Hf^{-2}f'$.

- In particular, on $H = 0$, we get $(f^{-1}H)' = f^{-1}H'$ and the null trajectories (not their parametrization) are the same.
- A second key fact is that the canonical one-form $\alpha = p(\theta)$ of $\mathcal{T}^*\mathcal{M}$, where θ is the canonical one-form of $\mathcal{M}$, passes down to the space of unparametrized geodesics, up to scale, giving that space a contact structure.

The dimension of the space of null geodesics

Through each point of space-time there is an $(n-2)$-dimensional family of null geodesics, specified by a future pointing null tangent direction at that point.

- The space of such directions is a sphere $\mathbb{S}^{n-2}$ of $n-2$ dimensions.
- Then the space of null geodesics is represented by placing an initial point on, say, a initial $(n-1)$-dimensional Cauchy surface and then choosing its direction, giving $(n-1)+(n-2)=2n-3$ degrees of freedom.
- This is always odd, as is appropriate for a contact manifold.

Null geodesics in Minkowski space-time

Minkowski space-time, denoted $\mathbb{M}_\infty$ is an affine space; in affine co-ordinates, the metric is just a fixed quadratic form on the n-dimensional tangent space. The null geodesics are then the straight lines with null tangent direction at every point.

- *Four dimensional Minkowski space-time is special!*

In *any dimension* $n \geq 3$, the space of null geodesics at a point is a $(n-2)$-sphere with a natural conformal structure, induced by restricting the metric to the null tangent vectors at that point.

- In dimension four, the space of null geodesics through any point is a two-sphere. This two-sphere then has a natural conformal structure, so also a natural complex structure, so is a *Riemann sphere*.
- Then the space of null geodesics, denoted $\mathbb{PN}_\infty$, has a unique Cauchy-Riemann structure, such that every point of Minkowski space-time acquires its natural Riemann sphere structure from the Cauchy-Riemann structure.

The five dimensional hyperquadric

Twistor theory precisely describes and encapsulates this hyperquadric.

- Twistor space, $\mathbb{T}$, is a four-dimensional complex vector space, equipped with a pseudo-hermitian form of signature $(2, 2)$.
- The space $\mathbb{PN}$ is the five-dimensional space of *one-dimensional subspaces* of $\mathbb{T}$ that are totally null.
- Conformally compactified Minkowski space-time, denoted $\mathbb{M}$, is the four-dimensional space of *two-dimensional subspaces* of $\mathbb{T}$ that are totally null.
- Then $\mathbb{PN}$ is naturally the space of null geodesics of $\mathbb{M}$.
- Given an element Z of $\mathbb{PN}$, the corresponding null geodesic is the set of all totally null two-dimensional subspaces of $\mathbb{T}$ that contain Z as a subspace.
- Distinct points X and Y of $\mathbb{M}$ are connected by a null geodesic $Z \in \mathbb{PN}$ if and only if $Z = X \cap Y$.

The embedding

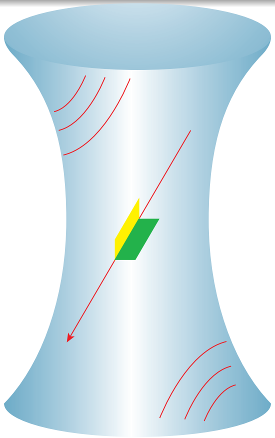
- The space $\mathbb{PT}$, called projective twistor space is the space of *one-dimensional subspaces* of $\mathbb{T}$.
- $\mathbb{PT}$ is a complex projective three-space, so in particular is a complex manifold of real dimension six.
- The natural embedding of $\mathbb{PN}$ as an hypersurface in $\mathbb{PT}$ gives $\mathbb{PT}$ its natural complex structure.
- Then $\mathbb{PN}$ is the hyperquadric, the "flat model" for Cauchy-Riemann geometry with a non-degenerate Levi-form of signature $(1, 1)$.
- Correspondingly, $\mathbb{M}$, embeds into compact complex Minkowski space-time, denoted $\mathbb{K}$, the Grassmannian of all two-planes in $\mathbb{T}$. This space is the Klein quadric, a non-singular quadric in complex projective five-space and has a natural complex conformally flat conformal structure, as does any non-singular quadric. Then the embedding of $\mathbb{M}$ into $\mathbb{K}$ is conformal. Here $\mathbb{K}$ is a complexification of $\mathbb{M}$.

The quadric of Oskar Klein

The Klein quadric $\mathbb{K}$ is the space of all two-dimensional subspaces of $\mathbb{T}$, so also the space of all projective lines in $\mathbb{PT}$.

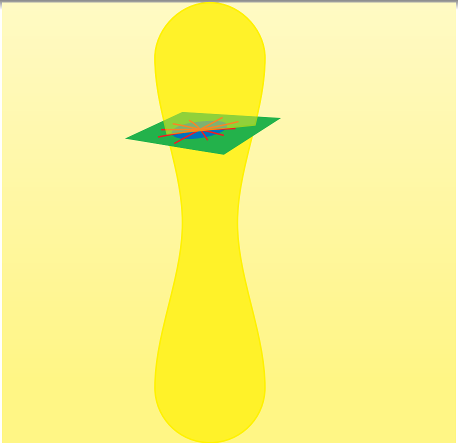
- If $\mathbb{X}$ and $\mathbb{Y}$ are distinct points of $\mathbb{K}$, they are null related if and only if $\mathbb{X} \cap \mathbb{Y}$ is one-dimensional, if and only if $\mathbb{X} + \mathbb{Y} \neq \mathbb{T}$. This gives $\mathbb{K}$ a natural conformally flat conformal structure.
- If $\mathbb{X} \in \mathbb{K}$, then $\mathbb{X} \wedge \mathbb{X}$ is a one-dimensional subspace of the six-dimensional space $\Omega^2(\mathbb{T})$, so a point in the five dimensional projective space $\mathbb{P}\Omega^2(\mathbb{T})$.
- The embedding $\mathbb{X} \in \mathbb{K} \to \mathbb{Y} = \mathbb{X} \wedge \mathbb{X} \in \mathbb{P}\Omega^2(\mathbb{T})$ has image a non-singular quadric in $\mathbb{P}\Omega^2(\mathbb{T})$, with the projective equation $\mathbb{Y} \wedge \mathbb{Y} = 0$, for $\mathbb{Y} \in \Omega^2(\mathbb{T})$.
  Hence the name the *Klein quadric!*
- Given a point $\mathbb{Y}$ of the Klein quadric, the space $\mathbb{X}$ such that $\mathbb{Y} = \mathbb{X} \wedge \mathbb{X}$, is the space of all $\mathbb{Z} \in \mathbb{T}$, such that $\mathbb{Z} \wedge \mathbb{Y} = 0$.

The Klein quadric



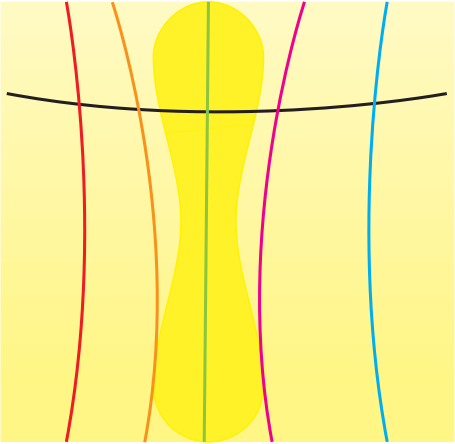
The Klein quadric featuring a null geodesic, the intersection of a pair of planes, a twistor plane and a dual twistor plane.

The twistor hyper-quadric



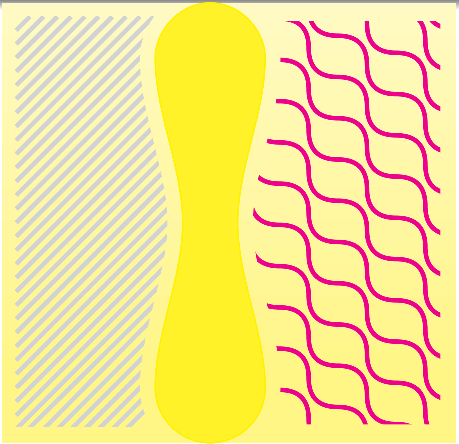
The null projective twistor space as a hyper-quadric in complex projective three-space.

Real and complex points



The various kinds of projective lines in projective twistor space.

Positive and negative frequency



The definition of massless quantum particles relies on fields that propagate into one-half of twistor space.

Recovering Minkowski: symmetry breaking

The natural symmetry group of $\mathbb{T}$, denoted $\mathbb{U}(\mathbb{T})$ is the pseudo-unitary group, isomorphic to $\mathbb{U}(2,2)$, consisting of all automorphisms of the complex vector space $\mathbb{T}$, preserving its pseudo-hermitian structure.

- $\mathbb{U}(\mathbb{T})$ is a real semi-simple Lie group of sixteen dimensions.
- $\mathbb{U}(\mathbb{T})$ acts on $\mathbb{M}$ as the conformal group (dimension fifteen).

We recover Minkowski space-time by breaking the symmetry.

- We select a point, denoted ∞, of $\mathbb{M}$.
- Then Minkowski space-time $\mathbb{M}_\infty$ is by definition the complement of the null cone of ∞, so is all $\mathbb{X} \in \mathbb{M}$, such that $\mathbb{X} + \infty = \mathbb{T}$.
- The symmetry group of $\mathbb{M}_\infty$ is reduced to the Weyl group (of real dimension eleven), the Poincaré group together with dilations.

The null geodesics of Schwarzschild

The Schwarzschild metric, which physically represents the gravitational field in the vacuum outside an isolated, static, spherically symmetric body, of mass m, can be written:

$$g = r^2(g_1 - g_2),$$

$$g_1 = \frac{(r-2m)}{r^3}\left(dt^2 - \frac{r^2 dr^2}{(r-2m)^2}\right), \quad g_2 = d\theta^2 + \sin^2(\theta) d\phi^2.$$

Here g_2 is the metric of a unit sphere, the co-ordinates (θ, ϕ) being the standard Euler angles.

Also, when $r > 2m$, t is a time variable and r a radial distance.

The null geodesics of Schwarzschild may be described as a pair of geodesics, sharing the same affine parameter, one in the two-space with metric g_1 and the other in the two-space with metric g_2 (so the latter curve is just a great circle on the sphere).

The elliptic curve

By symmetry, we may restrict to the equatorial plane, where $\theta = \frac{\pi}{2}$. Then the projection of the null geodesic into three dimensions is given by a relation between the variables r and ϕ:

$$\phi = \int \frac{dr}{\sqrt{L^2 r^4 - r^2 + 2mr}} = -\int \frac{du}{\sqrt{L^2 - u^2 + 2mu^3}}, \quad u = r^{-1}.$$

Here L is a real constant. When $L \neq 0$ and $27L^2m^2 \neq 1$, this is an elliptic integral. It follows that the generic null geodesics of Schwarzshild naturally complexify to give a curve on a torus, a complex curve of genus one: $y^2 = L^2r^4 - r^2 + 2mr$.
There are many kinds of null geodesics. If we focus on the case that the light ray comes in from infinity, reaches a closest point to the origin and then escapes to infinity again, then the scattering angle is given by a period of the torus, the integral over a closed curve beginning and ending with infinite r and passing through the branch point, when r reaches its minimum value at the largest real root of the cubic $L^2r^3 - r + 2m = 0$.

Null geodesics in general: Penrose limits

A null geodesic in an arbitrary space-time at first sight has no structure, except that it is a certain kind of curve. However it can be shown rather easily that it has a *natural projective structure*. Apart from this nothing.

- To see more structure, we have to look not just at the null geodesic itself, but at its immediate neighborhood.
- This will build in the key idea of null geodesic deviation.
- Fortunately, Roger Penrose has shown us how to do this: in the high-velocity limit, essentially obtained by "chasing the null geodesic", the space-time around the null geodesic takes a very specific form, called a Rosen plane wave.
- Such metrics are characterized by the fact that they have an homothety, a smooth dilation, fixing a null geodesic.
- These waves originate in the work of Einstein, Brinkmann, Baldwin, Jeffery and Rosen.
- We take these to constitute our moduli space.

Brinkmann-Baldwin-Jeffery-Einstein-Rosen waves

Our moduli space is the ensemble of metrics of the form:

$$G = dudv + p(u)(x,x)du^2 + g(u)(dx,dx).$$

Here u and v are real variables; $x \in \mathbb{V}$, a real vector space of dimension n, so our space-time has dimension $n+2$ (where, for us, $n > 0$).

- The symmetric tensors $p(u)$ and $g(u)$ depend smoothly on u, with $g(u)$ invertible.
- The space-time metric G has signature $(1+r, 1+s)$, when the tensor $g(u)$ has signature (r,s), with $r+s=n$. In particular, G is Lorentzian if and only if g is definite.
- On rescaling by a function $a(u) \neq 0$, putting $dU = a(u)du$, we get $a(u)G = dUdv + q(U)(x,x)dU^2 + h(U)(dx,dx)$, where $q(U)dU = p(u)du$ and $h^{-1}(U)dU = g^{-1}(u)du$.
- So our class of metrics is conformally invariant under such restricted conformal transformations.

The Brinkmann, Rosen and Kapadia forms

By suitable co-ordinate transformations, of the type:

$$(u, x, v) \to (u, y, w), \quad y = L(u)(x), \quad w = v - q(u)(x, x),$$

where $L(u)$ is linear and $q(u)$ is a symmetric tensor, we can:

- On the one hand reduce $p(u)$ to zero.
  This gives the Rosen plane wave metric:

  $$G = dudv + g(u)(dx, dx), \quad \det g(u) \neq 0.$$

- On the other hand reduce $g(u)$ to a constant invertible matrix. This gives the Brinkmann form of the metric:

  $$G = dudv + p(u)(x, x)du^2 + g(dx, dx).$$

  Here g is a constant invertible symmetric bilinear form.
- On the third hand, reduce $g(u)$ to diagonal form and simultaneously reduce the diagonal terms of $p(u)$ to zero.
  This gives the Kapadia form.

All these metrics depend on $\frac{1}{2}n(n+1)$ smooth functions of u.

The Schwarzian: a reminder

If $f(u)$ is a three-times differentiable function, its Schwarzian, denoted $S(f)$ is given by the formula:

$$4(f')^2 S(f) = 2f'f''' - 3(f'')^2.$$

Here a prime denotes differentiation with respect to u.

The Schwarzian has many nice elementary properties:

- $S(f) = 0$ iff f is fractional linear: $(cu + d)f(u) = au + b$, for some constants (a, b, c, d) with $ad - bc \neq 0$.
- If g is fractional linear, then $S(g \circ f) = S(f)$.
- If x_1 and x_2 are linearly independent solutions of the linear differential equation $x'' = q(u)x$, then $S(x_1 x_2^{-1}) = q$.
- $(S(f \circ g) - S(g))du^2 = (S(f) \circ g)dg^2$.
- If g is invertible: $(S(f) - S(g))du^2 = (S(f \circ g^{-1}) \circ g)dg^2$. Then $S(f) = S(g)$ if and only if $f \circ g^{-1}$ is fractional linear.

The diagonal Rosen metrics

The diagonal Rosen metrics are hypothesized by various theorists to play a dominant role near some singularities. We write the diagonal Rosen metrics in the following form:

$$G = f'(u)(dudv + h'(u)^{-1}(dx, dx)).$$

A prime denotes a derivative with respect to u; $f'(u) \neq 0$ is a conformal factor; $h(u)$ is a diagonal symmetric matrix, with invertible derivative. The covariant Riemann curvature tensor $\mathcal{R}$ of G has the remarkably simple form: $\mathcal{R} = -du \wedge \mathcal{S} \wedge du$, where $\mathcal{S}$ is the diagonal matrix: $\mathcal{S} = f'(S(h) - S(f)I)(h')^{-1}$. Here I is the identity matrix and $S(h)$ is the diagonal matrix, whose k-th diagonal entry is the Schwarzian of the k-th diagonal entry of h. The space-time is conformally flat if and only if $S(h)$ is pure trace if and only if each pair of diagonal entries of h are related by a fractional linear transformation. The Ricci tensor has at most one non-zero component, so the space-time is always locally conformal to vacuum.

The diagonal plane wave in four dimensions

In four dimensions, the diagonal plane wave metric is:

$$G = f'(u)\left(dudv + \frac{dx^2}{a'(u)} + \frac{dy^2}{c'(u)}\right).$$

- The covariant Riemann tensor is: $\mathcal{R} = du \wedge \mathcal{S} \wedge du$, with:

$$\mathcal{S} = f'\left((S(a) - S(f))\frac{dx^2}{a'(u)} + (S(c) - S(f))\frac{dy^2}{c'(u)}\right).$$

- The Einstein tensor is $(S(a) + S(c) - 2S(f))du^2$.
- The vacuum condition is $S(a) + S(c) = 2S(f)$.
- The conformal curvature is $-du \wedge \mathcal{C} \wedge du$, where we have:

$$2\mathcal{C} = f'(S(a) - S(c))\left(\frac{dx^2}{a'(u)} - \frac{dy^2}{c'(u)}\right).$$

- In particular the space-time is conformally flat if and only if $S(a) - S(c)$ vanishes if and only if $a \circ c^{-1}$ is fractional linear.

The general Kapadia metric in four dimensions

In four dimensions, we can write the general Kapadia metric as:

$$G = f'(u)\left(dudv - 2b(u)xydu^2 + \frac{dx^2}{a'(u)} + \frac{dy^2}{c'(u)}\right).$$

Remarkably, the Einstein tensor is the same as in the diagonal case, namely $(S(a) + S(c) - 2S(f))du^2$.
The Weyl tensor is now $-du \wedge \mathcal{C} \wedge du$, where we have:

$$\mathcal{C} = \frac{f'}{2}\left((S(a) - S(c))\left(\frac{dx^2}{a'} - \frac{dy^2}{c'}\right) + 2b(u)dxdy\right).$$

Now the metric is conformally flat if and only if both $a \circ c^{-1}$ is fractional linear and b vanishes identically. Note that in four dimensions, this is the fully general metric in our moduli space.

The connection with Universal Teichmüller space

We see that naturally occurring in connection with Rosen space-times is are diffeomorphisms from the real line to itself.

- Particularly in connection with the conformal structure, we see the emergence of relative diffeomorphisms and their Schwarzians.
- These are exactly the features of Universal Teichmüller space, which in its simplest form is concerned with diffeomorphisms from the circle to itself and the idea of conformal welding: gluing the complex upper half-plane to the lower by such a diffeomorphism of the boundary circle, so as to produce a Riemann surface.
- Here, in general, we would not expect necessarily to have circular null geodesics, but otherwise the structure is present. The Schwarzians then arise naturally in the construction of quadratic differentials on the Riemann surfaces.

The geodesic flow of the Rosen space-times

The Rosen plane wave metric is $G = dudv + g(u)(dx, dx)$, with u and v real and $x \in \mathbb{V}$, a real vector space of dimension at least one.

- The contact one-form is $\alpha = rdu + sdv + p(dx)$, with r and s real and $p \in \mathbb{V}^*$.
- The non-trivial Poisson brackets are: $\{r, u\} = \{s, v\} = 1$ and $\{p, \otimes x\} = \delta$, the Kronecker delta tensor of $\mathbb{V}$.
- The Hamiltonian is $H = 2rs + 2^{-1}g^{-1}(u)(p, p)$.
- Hamilton's equations are:

$$\dot{r} = \{H, r\} = -2^{-1}(g^{-1})'(p, p), \ \ \dot{s} = \{H, s\} = 0, \ \ \dot{p} = \{H, p\} = 0,$$

$$\dot{u} = \{H, u\} = 2s, \ \ \dot{v} = \{H, v\} = 2r, \ \ \dot{x} = \{H, x\} = (g^{-1})(p,).$$

The null geodesics of the Rosen space-times

Let h be a symmetric tensor, such that $h' = g^{-1}$.
Generically the constant s is non-zero, so since $\dot{u} = 2s$, we may use the variable u to parametrize the geodesic.
Also put $p = 2sq$, and integrate, giving:

$$x' = g^{-1}(q,), \quad x - x_0 = (h(u) - h(u_0))(q,),$$

$$v' = rs^{-1} = -g^{-1}(q, q), \quad v - v_0 = -(h(u) - h(u_0))(q, q).$$

Eliminating the vector q, we get an explicit formula for the equation of the null cone through an arbitrary point (u_0, x_0, v_0):

$$v - v_0 = -(h(u) - h(u_0))^{-1}(x - x_0, x - x_0).$$

Note that in particular, the lines $v = v_0$, $x = x_0$, u arbitrary, are always null geodesics, for any real v_0 and x_0.

Null geodesics with $s = 0$

The null geodesics with $s = 0$, equivalently those with u constant (since $\dot{u} = 2s$) have u, p and s constant and:

$$\dot{r} = \{H, r\} = -2^{-1}(g^{-1})'(u_0)(p, p), \quad \dot{v} = \{H, v\} = 2r,$$
$$\dot{x} = \{H, x\} = (g^{-1})(u_0)(p,), \quad 0 = g^{-1}(u_0)(p, p),$$
$$r = r_0 - 2^{-1}z(g^{-1})'(u_0)(p, p), \quad v = v_0 + 2r_0 z - 2^{-1}z^2(g^{-1})'(u_0)(p, p),$$
$$x - x_0 = zg^{-1}(u_0)(p,), \quad g(u_0)(x - x_0, x - x_0) = 0.$$

For the Lorentzian metrics, g^{-1} is definite, so $p = 0$, $r = r_0$, $x = x_0$ and v is arbitrary (unless $r_0 = 0$, in which case the geodesic is just a point), giving a single line through any given point. Then for each fixed u, there are an n-dimensional family of such lines, foliating the null hypersurface given by taking u constant. In the flat case these lines form the null geodesics emerging form the vertex of the null cone of the point at infinity represented by the null hyperplane with constant u. As u varies, these vertices themselves form a null geodesic at infinity.

Symmetries and conformal symmetries for Rosen

For the general Rosen metric, we can write down $(2n+1)$ independent symmetry vector fields:

$$H = \partial_v, \quad P = \partial_x, \quad Q = x\partial_v - 2^{-1}h(\partial_x), \quad h' = g^{-1}.$$

These obey the real *Heisenberg Lie algebra, $\mathfrak{h}_n$, for n degrees of freedom*:

$$[P, \otimes P] = 0, \ [Q, \otimes Q] = 0, \ [H, P] = 0, \ [H, Q] = 0, \ [P, \otimes Q] = \delta H.$$

When the metric is not flat, these are the only symmetries, except in two special cases, where there is one extra symmetry. Generically, there is *exactly one additional independent conformal symmetry*, the homothety $D = 2v\partial_v + x(\partial_x)$, which obeys the relations $[D, P] = -P, \ [D, Q] = -Q$ and $[D, H] = -2H$, so gives a natural grading of the Heisenberg algebra. Note that the *dilation vanishes precisely on the null geodesic, $x = 0, v = 0$, u arbitrary*.

The symmetry group action on null geodesics

The symmetry group of the generic Rosen space-time is the real Heisenberg group, $\mathfrak{H}_n$, for n degrees of freedom, which has dimension $(2n + 1)$.

- This is the *same* as the dimension of the space of null geodesics.
- The symmetries do not change the variable u, so the special null geodesics with u constant are mapped into themselves. In the Lorentzian case of interest, the null geodesics with u fixed form an homogeneous space for the group $\mathfrak{H}_n$, so the isotropy group is $n + 1$ dimensional.
- Remarkably the action on the remaining geodesics, so those with u varying is *transitive*.

Penrose limits and the theorem of Alekseevski

We have noted above that the Rosen space-times all have a homothety or dilation, a conformal Killing vector that reproduces the metric up to a constant scaling.
Also this dilation leaves invariant a null geodesic.
This is no accident, by a theorem of Alekseevski:

Theorem

If a space-time admits a smooth dilation group, leaving invariant a null geodesic, then the space-time is Rosen.

Penrose obtains his limiting space-times by creating a suitable dilation around a null geodesic, so following Alekseevski, we should not be surprised that the Penrose limit is Rosen!

Penrose's construction

Let a spacetime $\mathcal{M}$ be given with Lorentzian metric g and in it a null geodesic, denoted $\mathcal{L}$.

- Embed $\mathcal{L}$ locally in a non-rotating congruence of null geodesics, so $\mathcal{L}$ is an integral curve of an hypersurface orthogonal vector field l^a (we use abstract indices in the style of Penrose).
- This can always be done, locally, for example, by picking a timelike world-line through a point of $\mathcal{L}$ and forming the congruence by taking the generators of the future null cones of the points of the world-line.
- Choose co-ordinates, (u, v, x^i), such that $l = \partial_u$ and $l_a = m\partial_a v$, for some non-zero function m. So $g(l, l) = 0$ and $l^a \partial_a l_b = k l_b$, for some scalar k, where ∂_a is the Levi-Civita connection of g, so the metric has the form:

$$g = 2m(u, v, x^i)(du)(dv) + \beta(u, v, x^i, dv, dx^j).$$

The grading

Finally after replacing u by a suitable function $U(u, v, x^j)$, if necessary, we can arrange that the vector field ∂_v be null, so the metric takes the form:

$$g = 2m(u, v, x)(du)(dv) + 2b(u, v, x)(dx)(dv) + c(u, v, x)(dx, dx).$$

Here, we may regard x as a vector in a vector space $\mathbb{V}$ of n dimensions. Then b takes values in $\mathbb{V}^*$, whereas c is a symmetric tensor, taking values in $(\mathbb{V}^*)^2$.
Also, at worst after translating co-ordinates, we may assume, without loss of generality that our original null geodesic $\mathcal{L}$ is given by the equations $v = 0$ and $x = 0$, with u arbitrary.

- Now we introduce a *grading* on smooth functions and their differentials, assigning weights zero to u, one to x and two to v, effectively grading the metric in terms of its Taylor series expansion around the null geodesic given by $v = 0$ and $x = 0$.

Going to the limit

Then the least weight possible weight of the terms of g is two and we have an expansion: $g = g_2 + g_3 + g_4 + \ldots$, where g_k has integer grade k, for each $k \geq 2$. In particular, for the terms g_2, of lowest weight, we have the expression:

$$g_2 = 2m(u, 0, 0)(du)(dv) + c(u, 0, 0)(dx, dx).$$

Then g_2 is the Penrose limit and, by inspection, is a Rosen plane-wave space-time. We can formalize this as a limit by invoking the grading vector field $D = 2v\partial_v + x(\partial_x)$, which leaves the null geodesic $\mathcal{L}$ invariant. We have:

$$\exp(tD)(g_k) = t^k g_k, \lim_{t \to 0^+} t^{-2} \exp(tD)g = g_2.$$

Geometrically, we are focussing on a smaller and smaller tubular neighborhood around the null geodesic, blowing up until all features of the metric tensor g_{ab} are lost except its values on the null geodesic $\mathcal{L}$ itself.

The ambiguity in the Penrose limit

We have the following theorem, describing the ambiguity in the various ways we can construct a Penrose limit for a given null goedesic:

Theorem

Any two Penrose limits of a space-time along the same null geodesic are related by an explicit diffeomorphism.
The set of Penrose limits coming from the same geodesic is an orbit of the Heisenberg group of two degrees of freeodm
In the standard Rosen co-ordinate system, the infinitesimal generators of the group are the vector fields:

$$P = v\partial_x, \;\; Q = x\partial_u, \;\; H = v\partial_u.$$

The information in the Penrose limit

The fact that the Penrose limit gives the Rosen form of the plane wave rather than say the Brinkmann form is to us very significant.

- The metric coefficients of the Rosen form are directly related to all quantities of interest: the description of the null geodesics, the action of the symmetries, etc.
- This is to be compared with the Brinkmann form, which requires the integration of equations of Riccati type to determine these quantities.
- In the Brinkmann form the metric coefficients themselves directly determine the Riemann tensor, whereas in the Rosen form the metric coefficients are *potentials* for these quantities.

The complex curve in the half-plane

In four dimensions, the Penrose limit metric for a null geodesic is written as $(du)(dv) + g(u)(dx, dx)$, where $g(u)$ is a smooth symmetric bilinear form on $\mathbb{V}$, a two dimensional real vector space.

- Our complex curve comes first from factorizing the quadratic $g(u)(dx, dx)$ as $g(u)(dx, dx) = |m(u)(dx)|^2$. Here $m(u)$ takes values in $\mathbb{V}^* \otimes_{\mathbb{R}} \mathbb{C}$, the complexification of the real dual space of $\mathbb{V}$.
- Taking a suitable real basis for $\mathbb{V}$, we can write $m(u)(dx) = p(u)ds + q(u)dt$, where s and t are the two real components of the vector x.
- Then the complex ratio $\lambda(u) = p(u) : q(u)$ is relevant, up to a real projective transformation. Here, since the metric $g(u)$ is definite, we have $p(u)\overline{q}(u) - q(u)\overline{p}(u) \neq 0$, so the imaginary part of $\lambda(u)$ has a fixed sign, so we obtain a curve either in the upper or lower half-plane.

The complexification of the null geodesic

To see how to complexify a given null geodesic in an invariant way, we find it convenient to introduce local twistor transport, in the style of Penrose.

- The local twistor bundle is a complex vector bundle $\mathcal{T}$ of complex fiber dimension four, equipped with a pseudo-hermitian structure of signature $(2, 2)$.
- The bundle $\mathcal{T}$ has a preferred two dimensional sub-bundle $\mathcal{X}$, which is totally null and is construed as representing the points of the space-time $\mathcal{M}$.
- Then there is a natural one-form on $\mathcal{M}$ taking values in $\mathcal{X}^* \otimes \mathcal{T}/\mathcal{X}$, given by the formula $\theta(X) = idX \mod \mathcal{X}$, for any local section X of $\mathcal{X}$.
- Then θ is required to have maximal rank, which gives $\mathcal{M}$ a canonical conformal structure, such that a tangent vector V is non-null if and only if $\theta(V)$ is invertible.

Null geodesics and local twistors

In the local twistor language, a null geodesic is a curve equipped with a one dimensional (totally null) sub-bundle $\mathbb{Z}$ of the bundle $\mathcal{X}$, such that $Z \wedge dZ = 0$, for any local section Z of $\mathcal{Z}$. The local twistor transport is integrable along the null geodesic. At each point of the curve, we consider the space of twistors W, such that W is orthogonal to Z. Then $W \wedge Z$ represents a complex point of the Klein quadric in the fiber at the point, unless W itself is null, in which case it represents a real point. Then ensemble of all such points in the fiber is a Riemann sphere, with a preferred real circle and a preferred real point on that circle representing $\mathcal{X}$ at that point. As we move up the null geodesic, we parallelly propagate W, using the local twistor connection, giving a natural mapping from the Riemann sphere at one point to that at another and the preferred real point moves around appropriately. In this way we immerse the real null geodesic in a Riemann sphere.

Null geodesic deviation

Underlying the Penrose limit are the famous Sachs equations, which describe null geodesic deviation in four dimensions. Sachs reduced these equations to the following:

$$\rho' + \rho^2 + |\sigma|^2 + \Phi_{00} = 0, \quad \sigma' + 2\rho\sigma + \Psi_0 = 0.$$

Here Ψ_0 is the component of the Weyl spinor aligned with the null geodesic and Φ_{00} is the component of the Ricci curvature in the direction of motion. These are exactly the pieces of the curvature tensor that are encoded in the Penrose limit. ρ is the (real) rotation of a pencil of light rays abreast with a given null geodesic and σ is the complex shear. We would expect that these equations would be given by propagation of some quantity up the null geodesic, but what? It cannot be local twistor transport, or its tensor equivalent, which is the Cartan conformal connection. Instead it emerges that the correct notion is *conformal Killing transport*.

Thank you!