Equatorial Orbits of Test Particles in a Conformastatic Background

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Magnetized conformastatic space-time

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  A.C. Gutiérrez-Piñeres,
  Gen. Rel. Grav. 47 (2015) no.5, 54
- *Variational thermodynamics of relativistic thin disks*
  A.C. Gutiérrez-Piñeres, C.S. Lopez-Monsalvo, H. Quevedo,
  Gen. Rel. Grav. 47 (2015) no.12, 144
- *Exact Relativistic Magnetized Haloes around Rotating Disks* A.C. Gutiérrez-Piñeres, A.J.S. Capistrano,
  Adv. Math. Phys. 2015 (2015) 916026
- *Conformastatic disk-haloes in Einstein-Maxwell gravity*
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  Phys. Rev. D93 (2016) no.12, 124009

Conserved quantities

$$\text{Energy} \quad E: \quad p_t = -mce^{2\phi}\dot{t} \equiv -\frac{E}{c},$$

$$\text{Angular Momentum} \quad L: \quad p_\varphi = mr^2e^{-2\phi}\dot{\varphi} + \frac{q}{c}A_\varphi \equiv L,$$

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$$-c^2e^{2\phi}\dot{t}^2 + e^{-2\phi}(\dot{r}^2 + \dot{z}^2 + r^2\dot{\varphi}^2) = -\Sigma = \begin{cases} 0, \ \text{null curves,} \\ -c^2, \text{time-like curves,} \\ c^2, \text{space-like curves.} \end{cases}$$

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$$\text{The equatorial plane}\,(z=0): \dot{r}^2 + \Phi = \frac{E^2}{m^2c^2},$$

$$\text{Effective potential}: \Phi(r) \equiv \frac{L^2}{m^2r^2}\left(1 - \frac{qA_\varphi}{Lc}\right)^2 e^{4\phi} + \Sigma e^{2\phi}.$$

Circular Orbits

Conditions for the occurrence of circular orbits are

$$\frac{d\Phi}{dr} = 0, \quad \Phi = \frac{E^2}{m^2c^2}. \tag{1}$$

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Angular momentum of the particle in circular motion:

$$L_{c\pm} = \frac{qA_{\varphi}}{c} + \frac{qrA_{\varphi,r}e^{\phi} \pm \sqrt{\left(qrA_{\varphi,r}e^{\phi}\right)^2 - 4\Sigma c^2 m^2 r^3 \phi_{,r}\left(2r\phi_{,r} - 1\right)}}{2ce^{\phi}\left(2r\phi_{,r} - 1\right)}. \tag{2}$$

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Energy of the particle in circular motion:

$$E^{(\pm)}{}_{c\pm}=\pm\, mce^{\phi}\left(\Sigma+\xi^{(\pm)}_{\ c}\right)^{1/2}, \tag{3}$$

where

$$\xi^{(\pm)}_{\ c}=\frac{\left[qrA_{\varphi,r}e^{\phi}\pm\sqrt{\left(qrA_{\varphi,r}e^{\phi}\right)^2-4\Sigma c^2m^2r^3\phi_{,r}\left(2r\phi_{,r}-1\right)}\right]^2}{4m^2c^2r^2\left(2r\phi_{,r}-1\right)^2}.$$

Orbit in which the particle is located at rest (r_r) as seen by an observer at infinity

$$L = 0, \quad \frac{d\Phi}{dr} = 0. \tag{4}$$

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For an orbit with a rest radius $r = r_r$, the energy of the particle is,

$$E^{(\pm)}{}_{r\pm} = \pm mce^{\phi}\left(\Sigma + \xi_r^{(\pm)}\right)^{1/2} \tag{5}$$

where

$$\xi^{(\pm)}{}_{r} = \frac{q^2 e^{2\phi}\left[rA_{\varphi,r} \pm \left(rA_{\varphi,r} + 2A_{\varphi}\left(2r\phi_{,r} - 1\right)\right)\right]^2}{4m^2c^2r^2\left(2r\phi_{,r} - 1\right)^2}. \tag{6}$$

Last stable circular orbit

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$$\begin{aligned} L^{\pm}_{lsco} \;=\;& \frac{qA_\varphi}{c} + \Bigg\{ qe^{\phi}\left(8r^2A_{\varphi,r}\phi^2_{,r} + r^2A_{\varphi,rr} - 4rA_{\varphi,r}\right) \\ & \pm\Bigg[q^2e^{2\phi}\left(8r^2A_{\varphi,r}\phi^2_{,r} + r^2A_{\varphi,rr} - 4rA_{\varphi,r}\right)^2 \\ & -4(q^2r^2A^2_{\varphi,r}e^{2\phi} + 2c^2\Sigma m^2r^4\phi^2_{,r} + c^2\Sigma m^2r^4\phi_{,rr}) \\ & \times(8r^2\phi^2_{,r} + 2r^2\phi_{,rr} - 8r\phi_{,r} + 3)\Bigg]^{1/2}\Bigg\} \\ & \times\left[2ce^{\phi}(8r^2\phi^2_{,r} + 2r^2\phi_{,rr} - 8r\phi_{,r} + 3)\right]^{-1} \end{aligned} \tag{8}$$

The field of a punctual mass in Einstein-Maxwell gravity

$$ds^2 = -c^2 e^{2\phi} dt^2 + e^{-2\phi}(dr^2 + dz^2 + r^2 d\varphi^2), \tag{9}$$

$$\begin{aligned}
R_{\alpha\beta} &- \frac{1}{2} g_{\alpha\beta} R = k_0 E_{\alpha\beta}, \quad \nabla_\beta F^{\alpha\beta} = 0, \qquad (10)\\
E_{\alpha\beta} &= \frac{1}{4\pi} \left\{ F_{\alpha\gamma} {F_\beta}^\gamma - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \right\},\\
F_{\alpha\beta} &= A_{\beta,\alpha} - A_{\alpha\beta}, \quad A_\alpha = (0, A_\varphi).
\end{aligned}$$

A solution of the Einstein Maxwell equations

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\phi(r, z) &= \phi[U(r, z)], && (11) \\
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$$\phi = -\ln(1-U), \quad (13)$$

$$A_\varphi(r,z) = \frac{c^2}{G^{1/2}}\int_0^r \tilde{r}U(\tilde{r},z)d\tilde{r}, \quad (14)$$

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$$B_r = \frac{c^2}{G^{1/2}} r U_{,r} \qquad B_z = \frac{c^2}{G^{1/2}} r U_{,z} \ . \qquad (15)$$

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$$U(r,z) = -\frac{GM}{c^2 R}, \quad R^2 = r^2 + z^2 \ , \tag{16}$$

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$$\Phi(r) = \frac{c^6 r^2 (Lc - q\sqrt{G}M)^2}{m^2 (c^2 r + GM)^4} + \frac{\Sigma c^4 r^2}{(c^2 r + GM)^2}. \tag{20}$$

Circular motion of a charged test particle

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The angular momentum for a circular orbit with radius r_c

$$L_{c\,\pm} = \frac{q\sqrt{G}M}{c} \mp \frac{(c^2 r_c + GM)m}{c^2}\sqrt{\frac{\Sigma GM}{c^2 r_c - GM}} \qquad (21)$$

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The energy for a circular orbit with radius r_c

$$E_{c\,\pm} = \pm\frac{mc^4}{(c^2 r_c + GM)}\sqrt{\frac{\Sigma r_c^3}{c^2 r_c - GM}} \qquad (22)$$

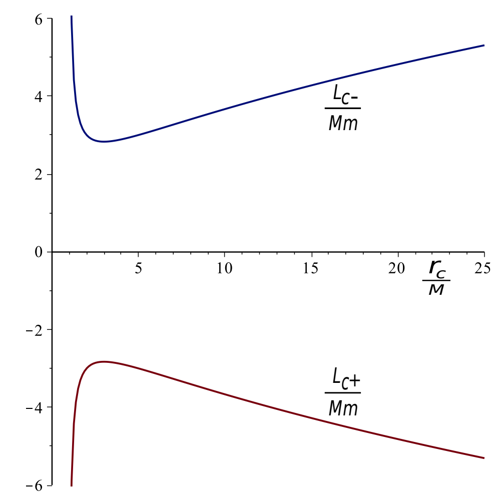

Figure: Angular momentum of a neutral test particle in terms of the radius orbit r_c/M

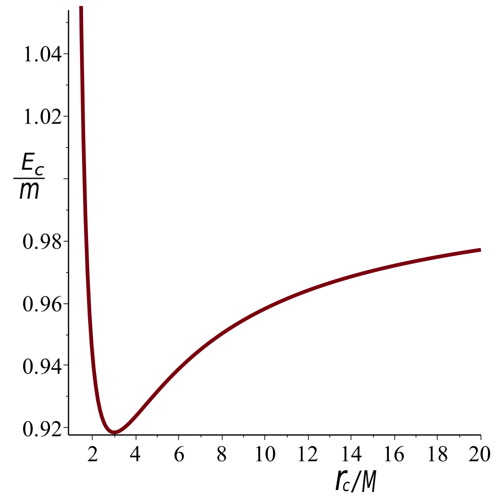

Figure: Energy of a charged test particle in terms of the radius orbit r_c/M

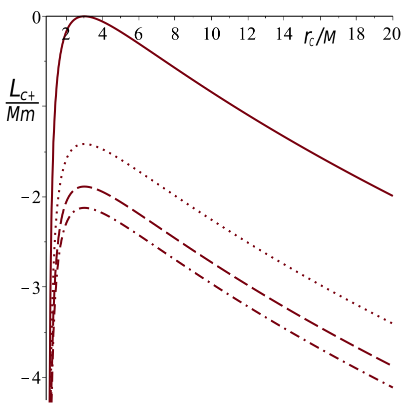

Figure: Angular momentum of charged particles in a time-like circular orbit In this graphic the angular momentum L_c^+/Mm is plotted as a function of the radius r_c/M for some values of q/m. The continuous curve corresponds to the value $q/m = 2\sqrt{2}$.

Charged test particle at rest $(L = 0)$ respect to an observer at infinity

$$\frac{r_{r\,\pm}}{M} = \frac{c^2q^2 - 2\Sigma G m^2 \pm \sqrt{c^2q^2\left(c^2q^2 - 8\Sigma G m^2\right)}}{2\Sigma m^2 c^2}\,, \qquad (23)$$

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$$\begin{aligned} E_{r+}^{(\pm)} &= +\frac{mc^2\sqrt{2q^2\left[q^2 - 2Gm^2 \pm \sqrt{q^2(q^2-8Gm^2)^2}\right]^3}}{\left[q^2 \pm \sqrt{q^2(q^2-8Gm^2)^2}\right]^2}\ , \\ E_{r-}^{(\pm)} &= -E_{r+}^{(\pm)}\ . \end{aligned} \tag{24}$$

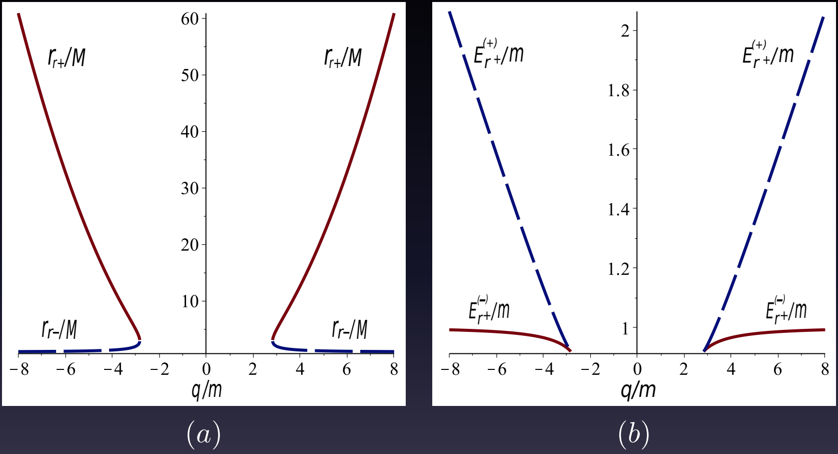

Figure: Radii and Energy of the charged particles for the time-like orbits characterized by the conditions $L = 0$ and $d\Phi/dr = 0$.

The last stable circular orbit

The angular momentum of a particle in the last stable circular orbit

$$L_{lsco\,\pm} = \frac{q\sqrt{G}M}{c} \pm \frac{2\sqrt{2\Sigma}GMm}{c^2}. \tag{25}$$

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The energy of a particle in the last stable circular orbit

$$E_{lsco\,\pm} = \pm\frac{3}{4}\sqrt{\frac{3\Sigma}{2}}mc\,. \tag{26}$$

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For a null curve

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If the charge of the particle is

$$q = 2\sqrt{2G}m, \qquad \text{then} \qquad L_{lsco-} = L_{c+} = L_r = 0. \quad (29)$$

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The last stable circular orbit occurs at the radius $r = 3GM/c^2$, independently of the value of the charge. Moreover, on the last stable orbit the particle is at rest, if the value of the charge is $q = \pm 2\sqrt{2G}m$.

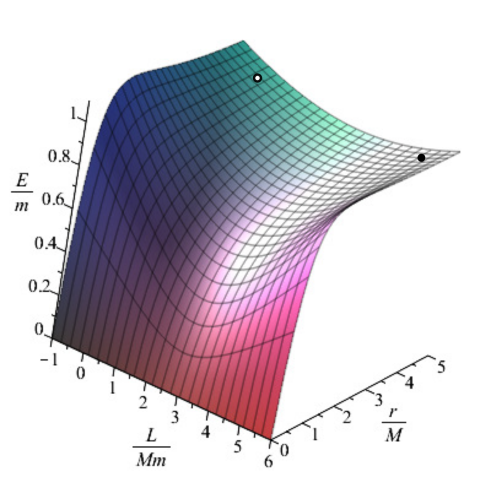

Figure: Particle at the last stable circular orbit, black point, ($q/m = 2\sqrt{2}, r/M = 3, L_{lsco+} = 4\sqrt{2}Mm$). Particle at rest, white point, ($q/m = 2\sqrt{2}, r/M = 3, L_{lsco-} = 0$).

Perihelion advance in a conformastatic magnetized spacetime

Value of the perihelion advance for a neutral test mass in the gravitational field of a punctual mass, endowed with a magnetic field.

$$\Delta\varphi = \frac{5\pi G M}{c^2 r_c} \ . \qquad (31)$$

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Value obtained for the Schwarzschild spacetime

$$\Delta\varphi = \frac{6\pi GM}{c^2 r_c}. \tag{32}$$

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Value obtained for the Schwarzschild spacetime

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The perihelion advance around a punctual magnetic mass is therefore always smaller than the value obtained in Einstein gravity alone. We conclude that the perihelion advance permits us to differentiate between a spherically symmetric mass and a conformally symmetric punctual magnetic mass.