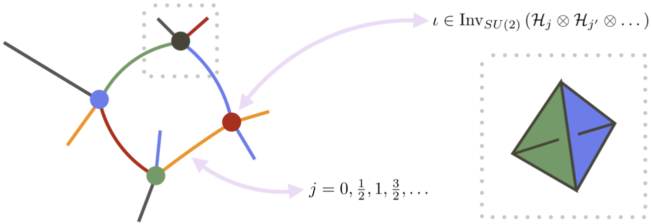
Angular Momentum, Centre of Mass, Spinors and Loop Quantum Gravity

Wolfgang Wieland
Perimeter Institute for Theoretical Physics

GR21

11 July 2016

Introduction

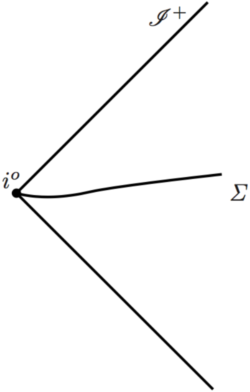


- Eigenstates of three-geometry are labelled by graphs Γ (combinatorial structure), coloured with $SU(2)$ spins $\vec{\jmath}$ and intertwiners $\vec{\iota}$.

$$\Psi = \sum_{\Gamma,\vec{\jmath},\vec{\iota}} \Psi_{\Gamma,\vec{\jmath},\vec{\iota}} \big|\Gamma, \vec{\jmath}, \vec{\iota}\big\rangle$$

- We do not measure microscopic spins and intertwiners, rather components of the Weyl tensor, multipoles of the gravitational field, mass, energy, angular momentum.
- We thus need a description to translate microscopic spins and intertwiners to macro-observables.

$$\big|\Gamma, \vec{\jmath}, \vec{\iota}\big\rangle \overset{?}{\longleftrightarrow} \big|M, J, \dots\big\rangle$$



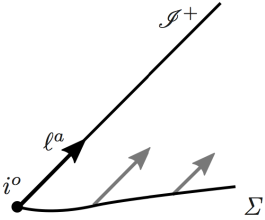
$$\big|\Gamma, \vec{\jmath}, \vec{\imath}\big\rangle \stackrel{?}{\longleftrightarrow} \big|M, J, \dots\big\rangle$$

There is a conceptual tension: Spins and intertwiners are defined locally, but the gravitational energy and angular momentum are two-surface integrals at spacelike infinity.

Strategy, step 1: Infinity is a mathematical idealisation — bring both energy and angular momentum back to finite distance — study quasilocal observables.

Step 2: Write the quasilocal observables in terms of LQG surface degrees of freedom — spinors.

Quasilocal gravitational charges from generalised Witten equations



- **energy and angular momentum** of an asymptotically flat spacetime are two-surface integrals over spacelike infinity

$$P^\alpha = \frac{1}{8\pi G}\lim_{r\to\infty}\int_{S_r^2} d^2\Omega\, r^3 n^a E_a{}^b \mathrm{d}x_b^\alpha$$

$$M^{\alpha\beta} = \frac{1}{8\pi G}\lim_{r\to\infty}\int_{S_r^2} d^2\Omega\, r^4 n^a B_a{}^b \partial_r^c \varepsilon^{de}{}_{bc} \mathrm{d}x_d^\alpha \mathrm{d}x_e^\beta$$

- **Parity conditions:** Leading r^{-3} order of $E_{ab} = C_{abcd}\partial_r^c\partial_r^d$ is parity even and leading r^{-4} order $B_{ab} = {}^*C_{abcd}\partial_r^c\partial_r^d$ is parity odd.
- **Positivity:** Given an asymptotically constant null-vector ℓ^a around spacelike infinity, Witten showed positivity of the energy $E_\ell = -P_\alpha \ell^\alpha$ by extending ℓ^a along an entire Cauchy hypersurface Σ and writing E_ℓ as a three-surface volume integral, which is manifestly positive.

Reminder: Witten's proof of positivity

- A spinor $\pi^A \in \mathbb{C}^2$ defines a future oriented null vector ℓ^a

$$\ell^a = \frac{\mathrm{i}}{\sqrt{2}} \sigma_{A\bar{A}}{}^{\mu} e_{\mu}{}^{a} \pi^A \bar{\pi}^{\bar{A}}$$

- Take an asymptotically constant spinor $\pi^A = {}^{\mathrm{o}}\pi^A + \mathcal{O}(r^{-1})$. Witten equation transports ${}^{\mathrm{o}}\pi^A$ from infinity along Σ into the bulk.

$$\frac{1}{2} \tilde{\varepsilon}^{abc} \Sigma^{A}{}_{Bab} D_c \pi^B = 0$$

- The ADM energy in the frame of ℓ^α can be written as the surface integral

$$\frac{1}{4\pi \mathrm{i} G} \int_{S \to \infty} \bar{\pi}^{\bar{A}} \sigma_{A\bar{A}} \wedge D\pi^A = P^\alpha \sigma^{A\bar{A}}{}_{\alpha}\, {}^{\mathrm{o}}\pi_A\, {}^{\mathrm{o}}\bar{\pi}_{\bar{A}}$$

- Positivity follows from Stoke's theorem and dominant energy condition

Conventions:

- Self-dual area two-form: $\Sigma_{AB} = \frac{1}{2\mathrm{i}} \sigma_{ABi} \big(\frac{1}{2} \epsilon_l{}^{i}{}_{m} e^l \wedge e^m + \mathrm{i} e^i \wedge e_0 \big)$
- Self-dual connection: $D_a \pi^A = \partial_a \pi^A + \frac{1}{2\mathrm{i}} \sigma^{A}{}_{Bi} \big(\Gamma^i{}_a + \mathrm{i} K^i{}_a \big) \pi^B$

Angular momentum and centre of mass

- A Lorentz transformation is generated by a vector field $V^\alpha = \Omega^\alpha{}_\beta X^\beta$, whose components grow linearly in r.
- The π spinors are asymptotically constant, to get an asymptotic Lorentz rotation, we once iterate the Witten equation [Witten, Penrose, Tod, Shaw,..., ww]

$$\frac{1}{2}\tilde{\varepsilon}^{abc}\Sigma^A{}_{Bab}\Big(D_c\omega^B + \sigma^B{}_{\bar{B}c}\bar{\pi}^{\bar{B}}\Big) = 0$$

- The integral over the corresponding Nester–Witten two-form returns the relativistic angular momentum $M_{\alpha\beta}$ shifted by the linear momentum P_α

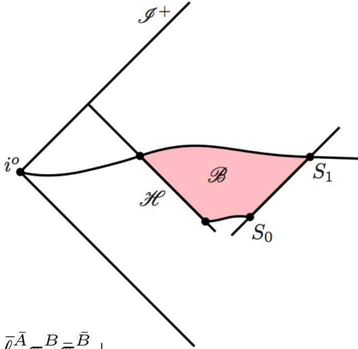
$$J_{\pi,\omega}[S_\infty] = \frac{1}{4\pi \mathrm{i} G}\int_{S\to\infty} \bar{\omega}^{\bar{A}}\sigma_{A\bar{A}} \wedge D\pi^A = P^\alpha \sigma^{A\bar{A}}{}_\alpha {}^{\mathrm{o}}\pi_A {}^{\mathrm{o}}\bar{\omega}_{\bar{A}} - M^{\alpha\beta}\Sigma^{AB}{}_{\alpha\beta} {}^{\mathrm{o}}\pi_A {}^{\mathrm{o}}\pi_B$$

- NB generalised Witten equations are well-defined on surfaces of arbitrary signature.

- We look at the dynamics of GR in a finite region $\mathcal{B}$. (observer + system observed)
- The spinors solve the Witten equation along the boundary of $\mathcal{B}$.
- Quasi-local energy

$$E_\pi[S] = \frac{1}{4\pi \mathrm{i} G} \int_S \bar{\pi}^{\bar{A}} \sigma_{A\bar{A}} \wedge D\pi^A$$

- The Einstein equations determine the change of $E_\pi[S]$ through



$$E_\pi[S_1] - E_\pi[S_0] = -4\pi G \int_{\mathcal{H}} \mathrm{d}u \wedge d^2 v\, T_{A\bar{A}B\bar{B}} \ell^A \bar{\ell}^{\bar{A}} \pi^B \bar{\pi}^{\bar{B}} +$$

$$+ \int_{\mathcal{H}} \mathrm{d}u \wedge d^2 v \Big[q^{ab} D_a \bar{\pi}^{\bar{A}} D_b \pi^A \bar{\ell}_{\bar{A}} \ell_A + \ell^a \ell^b D_a \bar{\pi}^{\bar{A}} D_b \pi^A \bar{k}_{\bar{A}} k_A \Big] \geq 0$$

- Analogous construction for the angular-momentum charges $J_{\pi,\omega}[S]$.

What does this have to do with LQG?

Spinors as LQG boundary degrees of freedom

- The expressions for quasi-local energy and angular momentum are two-dimensional surface integrals over a cross-section of the three-boundary. The dynamics is coded in the generalised Witten equations that determine balance laws between different cross-sections.
- The integrals for quasi-local energy and angular momentum can be derived from a Hamiltonian perspective through a (3,2)+1 split of a suitable bulk+boundary action (with boundary term $\propto \Sigma_{AB} \wedge \omega^A D\pi^B$).
- This nicely resonates with LQG: The theory is based on Wilson lines for the self-dual connection. A Wilson line ending at a two-surface generates a charge. The charge for an $SL(2,\mathbb{C})$ gauge connection is nothing but spin. Hence again spinors at the boundary.

Can we find a boundary action realising this principle?

- A physical motion extremizes the action

$$S_M[\Pi, A|\pi, \ell] = \int_M \Pi_{AB} \wedge F^{AB}[A] + \int_{\mathcal{H}} \pi_A \wedge D\ell^A + \mathrm{cc.}$$

- In the class of all fields satisfying the bulk and boundary constraints

$$\Pi_{AB} = \frac{\mathrm{i}}{8\pi\beta G}(\beta + \mathrm{i})\Sigma_{AB}, \quad \Sigma_{AB} = \frac{1}{2} e_{(A}{}^{\bar{C}} \wedge e_{B)\bar{C}}$$
$$\frac{\mathrm{i}}{\beta + \mathrm{i}} \pi_A \ell^A + \mathrm{cc.} = 0.$$

- Resulting EOM: Einstein equations in the bulk plus boundary conditions for non-expanding null surface. Generalisation to arbitrary null surfaces straight forward.

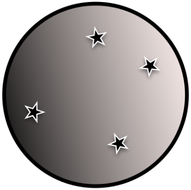
Relation to Loop Quantum Gravity

- Boundary symplectic structure for the spinor fields on the sphere:

$$\big\{\tilde{\pi}_A(p),\ell^A(q)\big\}=\delta^A_B\tilde{\delta}^{(2)}(p,q)$$

- State vectors are cylindrical functions:

$$\Psi_f[\ell]=f\big(\ell(q_1),\dots,\ell(q_N)\big)$$



- 2+1 split of the boundary action gives the boundary energy.
- Simplicity constraint

$$\tilde{C}=\frac{\mathrm{i}}{\beta+\mathrm{i}}\tilde{\pi}_A\ell^A+\mathrm{cc.}=0$$

- Imposing the simplicity constraint yields LQG Hilbert space. Quantisation of area

$$\mathrm{Ar}=8\pi\hbar\beta G/c^3\times j,\quad j=0,\tfrac{1}{2},1,\dots$$

Conclusions

Summary

- Spinors a LQG boundary DOF: Loop Gravity is based on an $SL(2,\mathbb{C})$ connection. A connection couples to flavour-charges at the boundary. The flavour of an $SL(2,C)$ connection is spin. In the piecewise-flat setting only the boundary term survives. Thus spinors as the fundamental DOF of LQG — a form of weak or local holography.
- Relation to earlier work on spinors in LQG: The symplectic structure for the boundary spinors is exactly the same that underlies the spinorial parametrisation of the LQG phase space $T^*SL(2,\mathbb{C})$ resp. $T^*SU(2)$ on a link [Freidel, Speziale, Livine, ww]. [see also recent paper by Bianchi, Yokomizo, Guglielmon, Hackl for further developments]
- Relation to quasi-local charges: The Witten equations are valid on surfaces of arbitrary signature and can be seen as equations of motion for the LQG boundary spinors along the entire three-boundary of a four-dimensional spacetime region. The corresponding canonical Hamiltonian is a two-surface integral over arbitrary two-dimensional cross sections of the three-boundary.