Non-OT G_2 spike solution

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Abstract

We generalize the orthogonally transitive (OT) G_2 spike solution to the non-OT G_2 case.
This is achieved by applying Geroch's transformation on a Kasner seed.
The new solution contains two more parameters than the OT G_2 spike solution.
Unlike the OT G_2 spike solution, the new solution always resolves its spike.

Classification of G_2 solutions

Consider vacuum solutions of Einstein's field equations that admit two Killing vector fields (KVFs) ξ and η.

The two KVFs form a G_2 group, which may be Abelian (iff the KVFs commute) or not.

A KVF ξ is said to be hypersurface-orthogonal (HO) if it satisfies

$$\xi_{[a;b}\xi_{c]} = 0.$$

A G_2 group is said to be orthogonally transitive (OT) if the two KVFs ξ and η satisfy

$$\xi_{[a;b}\xi_c\eta_{d]} = 0, \quad \eta_{[a;b}\eta_c\xi_{d]} = 0.$$

If one of the KVFs in an OT G_2 group is HO, then the other KVF is also HO, and such a group is called diagonal, because the metric for the space spanned by the KVFs can be made diagonal.

Wainwright's classification of G_2 solutions with spacelike KVFs

Wainwright 1981 classified G_2 solutions with spacelike KVFs as follows:

$G_2 I$: Abelian G_2
- Class A: G_2 is non-OT
  - Class A(i): no HO KVF
  - Class A(ii): one HO KVF
- Class B: G_2 is OT
  - Class B(i): no HO KVF
  - Class B(ii): two HO KVFs (i.e. diagonal)

$G_2 II$: Non-Abelian G_2
- Sintes 1996 similarly classified $G_2 II$.

Introduction

According to general relativity, in the asymptotic regime near spacelike singularities, a spacetime would oscillate between Kasner states. The BKL conjectures hold except where and when spikes occur. Spikes are a recurring inhomogeneous phenomenon in which the fabric of spacetime temporarily develops a spiky structure as the spacetime oscillates between Kasner states.

Previously, the orthogonally transitive (OT) G_2 spike solution, which is important in describing the recurring spike oscillation, was generated by applying the Rendall-Weaver transformation on a Kasner seed solution. The solution is unsatisfactory, however, in that it contains permanent spikes, and there is a debate whether permanent spike are actually unresolved spike transitions in the oscillatory regime or are really permanent. In other words, would the yet undiscovered non-OT G_2 spike solution contain permanent spikes? The proponents for permanent spikes argue that the spatial derivative terms of a permanent spike are negligible, and hence the spike stays permanent. The opponents base their argument on numerical evidence that the permanent spike is mapped by an R_1 frame transition to a regime where the spatial derivative terms are not neglibigle, which allows the spike to resolve.

Iwasawa frame

To settle the debate, we need to find the non-OT G_2 spike solution. It was found that Geroch's transformation would generate the desired solution, which always resolves its spike.

For our purpose, we express a metric g_{ab} using the Iwasawa frame, as follows. Indicies $0, 1, 2, 3$ corresponds to coordinates τ, x, y, z. Assume zero vorticity (zero shift). The metric components in terms of b's and n's are given by

$$
\begin{aligned}
g_{00} &= -N^2 \\
g_{11} &= e^{-2b_1}, \quad g_{12} = e^{-2b_1} n_1, \quad g_{13} = e^{-2b_1} n_2 \\
g_{22} &= e^{-2b_2} + e^{-2b_1} n_1^2, \quad g_{23} = e^{-2b_1} n_1 n_2 + e^{-2b_2} n_3 \\
g_{33} &= e^{-2b_3} + e^{-2b_1} n_2^2 + e^{-2b_2} n_3^2.
\end{aligned}
$$

One advantage of the Iwasawasa frame is that the determinant of the metric is given by

$$\det g_{ab} = -N^2 e^{-2b_1 - 2b_2 - 2b_3}.$$

Kasner metric

A pedagogical starting point is the Kasner solution with the following parametrization:

$$b_1 = \frac{1}{4}(w^2 - 1)\tau, \quad b_2 = \frac{1}{2}(w + 1)\tau, \quad b_3 = -\frac{1}{2}(w - 1)\tau,$$
$$N = -e^{-b_1 - b_2 - b_3} = -e^{-\frac{1}{4}(w^2+3)\tau},$$

and $n_1 = n_2 = n_3 = 0$.
We shall use a linear combination of all three KVFs

$$a_1\partial_x + a_2\partial_y + a_3\partial_z.$$

as the KVF in Geroch's transformation, so that the transformation generates the most general metric possible from the given seed.

Change of coordinates

To simplify the KVF before applying Geroch's transformation, make the coordinate change

$$x = X + n_{10} Y + n_{20} Z, \quad y = Y + n_{30} Z, \quad z = Z$$

where n_{10}, n_{20}, n_{30} are constants. Then the metric parameters b_1, b_2, b_3 and N are unchanged but $n_1 = n_{10}$, $n_2 = n_{20}$, $n_3 = n_{30}$ are now constants instead of zero. The KVF becomes

$$(a_3(n_{10} n_{30} - n_{20}) - a_2 n_{10} + a_1)\partial_X + (a_2 - a_3 n_{30})\partial_Y + a_3 \partial_Z .$$

We cannot set the Z component to zero, but we can set the X and Y components to zero, leading to

$$n_{30} = \frac{a_2}{a_3}, \quad n_{10} = \frac{a_1}{a_3}.$$

Without loss of generality, we set $a_3 = 1$, and so $n_{30} = a_2$ and $n_{10} = a_1$. n_{20} remains free. We will see later that it can be used to eliminate any y-dependence.

Change of coordinates

To make transparent the effect of Geroch's transformation on the b's, it is best to adapt the KVF to ∂_x. So we make another coordinate change to swap X and Z:

$$X = \tilde{z}, \quad Y = \tilde{y}, \quad Z = \tilde{x},$$

which in effect introduces frame rotations to the Kasner solution. The Kasner solution now has

$$\begin{aligned}
N &= -\mathrm{e}^{-\frac{1}{4}(w^2+3)\tau} \\
\mathrm{e}^{-2b_1} &= \mathrm{e}^{(w-1)\tau} + n_{20}^2 \mathrm{e}^{-\frac{1}{2}(w^2-1)\tau} + n_{30}^2 \mathrm{e}^{-(w+1)\tau} \\
\mathrm{e}^{-2b_2} &= \frac{\mathcal{A}^2}{\mathrm{e}^{-2b_1}} \\
\mathrm{e}^{-2b_3} &= \mathrm{e}^{-\frac{1}{2}(w^2+3)\tau} \mathcal{A}^{-2} \\
n_1 &= \frac{n_{30}\mathrm{e}^{-(w-1)\tau} + n_{10}n_{20}\mathrm{e}^{-\frac{1}{2}(w^2-1)\tau}}{\mathrm{e}^{-2b_1}} \\
n_2 &= \frac{n_{20}\mathrm{e}^{-\frac{1}{2}(w^2-1)\tau}}{\mathrm{e}^{-2b_1}} \\
n_3 &= \mathrm{e}^{-\frac{1}{2}(w^2-1)\tau} \mathcal{A}^{-2} \left[n_{30}(n_{10}n_{30} - n_{20})\mathrm{e}^{-(w+1)\tau} + n_{10}\mathrm{e}^{(w-1)\tau} \right],
\end{aligned}$$

Change of coordinates

where

$$\mathcal{A}^2 = (n_{10}n_{30} - n_{20})^2 \mathrm{e}^{-\frac{1}{2}(w+1)^2\tau} + n_{10}^2 \mathrm{e}^{-\frac{1}{2}(w-1)^2\tau} + \mathrm{e}^{-2\tau}.$$

Effectively, we are applying Geroch's transformation to the seed solution above, using the KVF $\partial_{\tilde{x}}$. We shall now drop the tilde from the coordinates.

Applying Geroch's transformation

Applying Geroch's transformation using a KVF ξ involves the following steps. First compute

$$\lambda = \xi^a \xi_a$$

and integrate the equation

$$\nabla_a \omega = \varepsilon_{abcd} \xi^b \nabla^c \xi^d$$

for the general solution for ω. ω is determined up to an additive constant ω_0. In our case we get

$$\lambda = \mathrm{e}^{-2b_1} = \mathrm{e}^{(w-1)\tau} + \mathrm{e}^{-\frac{1}{2}(w^2-1)\tau} n_{20}^2 + \mathrm{e}^{-(w+1)\tau} n_{30}^2, \quad \omega = 2wn_{30}z - Ky + \omega_0,$$

where the constant K is given by

$$K = \frac{1}{2}(w-1)(w+3)n_{20} - 2wn_{10}n_{30}.$$

We could absorb ω_0 by a translation in the z direction if $wn_{30} \neq 0$, but we shall keep ω_0 for the case $wn_{30} = 0$.

Applying Geroch's transformation

The next step involves finding a particular solution for α_a and β_a:

$$\nabla_{[a}\alpha_{b]} = \frac{1}{2}\varepsilon_{abcd}\nabla^c\xi^d, \quad \xi^a\alpha_a = \omega,$$
$$\nabla_{[a}\beta_{b]} = 2\lambda\nabla_a\xi_b + \omega\varepsilon_{abcd}\nabla^c\xi^d, \quad \xi^a\beta_a = \omega^2 + \lambda^2 - 1.$$

Without loss of generality, we choose $\theta = \frac{\pi}{2}$ in Geroch's transformation, so α_a is not needed in η_a below. We assume that β_a has zero τ-component. Its other components are

$$\beta_1 = \omega^2 + \lambda^2 - 1$$
$$\beta_2 = n_{10}n_{20}^3 e^{-(w^2-1)\tau} + \left[2\frac{w-1}{w+1}n_{10}n_{20}n_{30}^2 + \frac{4}{w+1}n_{20}^2 n_{30}\right] e^{-\frac{1}{2}(w+1)^2\tau}$$
$$+ 2\frac{w+1}{w-1}n_{10}n_{20}e^{-\frac{1}{2}(w-1)^2\tau} + (w+1)n_{30}e^{-2\tau} + n_{30}^3 e^{-2(w+1)\tau} + F_2(y,z)$$
$$\beta_3 = n_{20}^3 e^{-(w^2-1)\tau} + 2n_{20}n_{30}^2\frac{w-1}{w+1}e^{-\frac{1}{2}(w+1)^2\tau} + 2n_{20}\frac{w+1}{w-1}e^{-\frac{1}{2}(w-1)^2\tau} + F_3(y,z$$

where $F_2(y,z)$ and $F_3(y,z)$ satisfy the constraint equation

$$-\partial_z F_2 + \partial_y F_3 + 2(w-1)\omega = 0.$$

Applying Geroch's transformation

For our purpose, we want F_3 to be as simple as possible, so we choose

$$F_3 = 0, \quad F_2 = \int 2(w-1)\omega \mathrm{d}z = 2w(w-1)n_{30}z^2 - 2(w-1)Kyz + 2(w-1)\omega_0 z.$$

The last step constructs the new metric. Define $\tilde{\lambda}$ and η_a as

$$\frac{\lambda}{\tilde{\lambda}} = (\cos\theta - \omega\sin\theta)^2 + \lambda^2 \sin^2\theta,$$
$$\eta_a = \tilde{\lambda}^{-1}\xi_a + 2\alpha_a \cos\theta\sin\theta - \beta_a \sin^2\theta.$$

The new metric is given by

$$\tilde{g}_{ab} = \frac{\lambda}{\tilde{\lambda}}(g_{ab} - \lambda^{-1}\xi_a\xi_b) + \tilde{\lambda}\eta_a\eta_b.$$

In our case $\tilde{g}_{ab}$ is given by the metric parameters

$$\tilde{N}^2 = N^2(\omega^2 + \lambda^2)$$
$$e^{-2\tilde{b}_1} = \frac{e^{-2b_1}}{\omega^2 + \lambda^2}$$
$$e^{-2\tilde{b}_2} = e^{-2b_2}(\omega^2 + \lambda^2)$$
$$e^{-2\tilde{b}_3} = e^{-2b_3}(\omega^2 + \lambda^2)$$

Applying Geroch's transformation

$$
\begin{aligned}
\tilde{n}_1 &= -2w(w-1)n_{30}z^2 + 2(w-1)Kyz - 2(w-1)\omega_0 z \\
&\quad + \frac{\omega^2}{\lambda}\left(n_{30}\mathrm{e}^{-(w+1)\tau} + n_{10}n_{20}\mathrm{e}^{-\frac{1}{2}(w^2-1)\tau}\right) \\
&\quad - \left[n_{30}w\mathrm{e}^{-2\tau} + \frac{w+3}{w-1}n_{10}n_{20}\mathrm{e}^{-\frac{1}{2}(w-1)^2\tau}\right. \\
&\quad \left. + \frac{w-3}{w+1}n_{20}n_{30}(n_{10}n_{30}-n_{20})\mathrm{e}^{-\frac{1}{2}(w+1)^2\tau}\right] \\
\tilde{n}_2 &= n_{20}\mathrm{e}^{-\frac{1}{2}(w^2-1)\tau}\left[-\frac{w+3}{w-1}\mathrm{e}^{(w-1)\tau} - n_{30}^2\frac{w-3}{w+1}\mathrm{e}^{-(w+1)\tau} + \frac{\omega^2}{\lambda}\right] \\
\tilde{n}_3 &= \mathcal{A}^{-2}\left[n_{10}\mathrm{e}^{-\frac{1}{2}(w-1)^2\tau} + n_{30}(n_{10}n_{30}-n_{20})\mathrm{e}^{-\frac{1}{2}(w+1)^2\tau}\right],
\end{aligned}
$$

and $\mathcal{A}$ is the area density of the G_2 orbits.

The new metric

The new solution admits two commuting KVFs:

$$\partial_x, \quad [-(w-1)K^2y^2 + 2(w-1)K\omega_0 y]\partial_x + 2wn_{30}\partial_y + K\partial_z.$$

Their G_2 action is non-OT, unless $n_{10} = n_{20} = 0$. The solution is also the first non-OT Abelian G_2 explicit solution found.

In the next section we shall focus on the case where $K = 0$, or equivalently, where

$$n_{20} = \frac{4w}{(w-1)(w+3)} n_{10} n_{30},$$

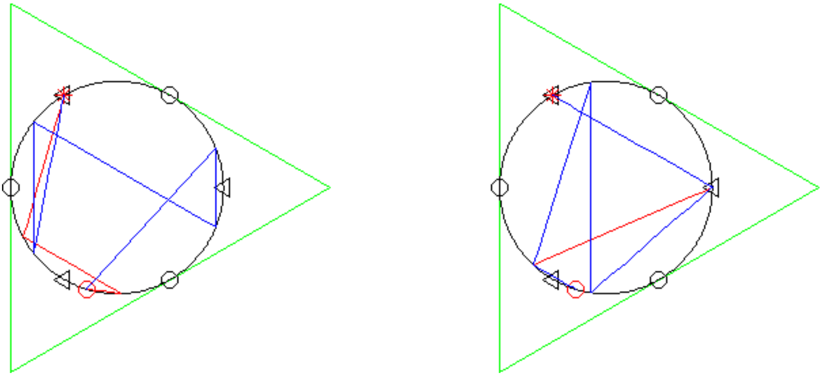
which turns off the R_2 frame transition and eliminates the y-dependence.

Special cases

The mixed frame/curvature transition $\mathcal{T}_{N_1 R_1}$ is described by the metric $\tilde{g}_{ab}$ with $n_{20} = n_{30} = 0$. Both the double frame transition and the mixed frame/curvature transition are encountered in the exceptional Bianchi type $\mathrm{VI}^*_{-1/9}$ cosmologies.
Setting $n_{10} = n_{20} = 0$ yields the OT G_2 spike solution.

The dynamics of the solution

To describe the dynamics of the non-OT spike solution, we shall plot the state space orbit projected onto the Hubble-normalized (Σ_+, Σ_-) plane.



Alternative spike orbits for $w = 5$. Left panel is the first alternative orbit, right panel is the second alternative. Spike orbits are in red, faraway orbits in blue. A red circle marks the start of the orbits, a red star marks the end.

The dynamics of the solution

The non-OT spike solution (with $K = 0$, $\omega_0 = 0$) goes from a Kasner state with $2 < w < 3$, through a few intermediate Kasner states, and arrives at the final Kasner state with $w < -1$. The transitions are composed of spike transitions and R_1 frame transitions. The non-OT spike solution always resolves its spike, unlike the OT spike solution with $|w| < 1$, which has a permanent spike.

For a typical Kasner source with $2 < w < 3$, there are six non-OT spike solutions, some of which are equivalent, that start there. For example, non-OT spike solutions with $|w| = \frac{1}{3},\ 2,\ 5$ all start at $w_{\text{source}} = \frac{7}{3}$. From there, however, there are two extreme alternative spike orbits. The first alternative is to form a "permanent" spike, followed by an R_1 transition, and lastly to resolve the spike. This alternative is more commonly encountered (assuming that permanent spikes are more commonly encountered than no-spike at the end of a Kasner era). The second alternative is to undergo an R_1 transition first, followed by a transient spike transition, and finish with another R_1 transition. By varying n_{10} and n_{30}, one can get orbits that are close to one extreme alternative or the other, or some indistinct mix.

Summary

We went through the steps of generating the non-OT G_2 spike solution, and illustrated its state space orbits for the case $K = 0$, which show two extreme alternative orbits. More importantly, the non-OT G_2 spike solution always resolves its spikes, in contrast to its OT G_2 special case which produces an unresolved permanent spike for some parameter values. The non-OT G_2 spike solution shows that, in the oscillatory regime near spacelike singularities, unresolved permanent spikes are artefacts of restricting oneself to the OT G_2 case, and that spikes are resolved in the more general non-OT G_2 case. Therefore spikes are expected to recur in the oscillatory regime rather than to become permanent spikes.

For the stiff fluid case, see Coley & Lim 2016.

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