Algebraic classification of spacetimes - recent developments

V. Pravda, Institute of Mathematics, CAS, Prague, CZ

July 12, 2016

Null frame

On a d-dimensional Lorentzian manifold we will work in the frame

$$\boldsymbol{n},\ \boldsymbol{\ell},\ \boldsymbol{m^{(i)}}, \quad i,j,k = 2 \dots d-1,$$

with two null vectors $\boldsymbol{n}$, $\boldsymbol{\ell}$

$$\ell^a \ell_a = n^a n_a = 0, \quad \ell^a n_a = 1, \quad a = 0 \dots d-1$$

and $d-2$ spacelike vectors

$$\boldsymbol{m^{(i)}}, \quad m^{(i)a} m^{(j)}_a = \delta_{ij}, \qquad i,j,k = 2 \dots d-1.$$

The **metric** has the form

$$g_{ab} = 2\ell_{(a} n_{b)} + \delta_{ij} m^{(i)}_a m^{(j)}_b .$$

Lorentz transformations

The group of ortochronous Lorentz transformations is generated by **null rotations**

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell} + z_i \boldsymbol{m}^{(i)} - \frac{1}{2} z^i z_i \, \boldsymbol{n}, \quad \hat{\boldsymbol{n}} = \boldsymbol{n}, \quad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)} - z_i \boldsymbol{n},$$

spins

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell}, \quad \hat{\boldsymbol{n}} = \boldsymbol{n}, \quad \hat{\boldsymbol{m}}^{(i)} = X^i{}_j \boldsymbol{m}^{(j)},$$

and **boosts**

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \quad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \quad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}.$$

A quantity q has a **boost weight** b if it transforms under a boost according to

$$\hat{q} = \lambda^b q.$$

Boost order

Boost order of a tensor $\mathbf{T}$ is the maximum boost weight of its frame components.

Proposition

Let $\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}^{(i)}$ and $\hat{\boldsymbol{\ell}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{m}}^{(i)}$ be two null-frames with $\boldsymbol{\ell}$ and $\hat{\boldsymbol{\ell}}$ scalar multiples of each other. Then, the boost order of a given tensor is the same relative to both frames.

Thus boost order of a tensor depends (only) on the choice of a null direction - $\boldsymbol{\ell}$.

Project the tensor $\boldsymbol{T}$ on the null frame and sort its components by their boost weight

$$\boldsymbol{T} = \sum_{b=b_{min}}^{b_{max}} (\boldsymbol{T})_{(b)}.$$

algebraic type	conditions
type G	general
type I	compts with $b = b_{max}$ can be set to zero
type II	compts with $b > 0$ can be set to zero
type III	compts with $b \geq 0$ can be set to zero
type N	compts with $b > b_{min}$ can be set to zero

Bivector

Example: for a bivector $K_{ab} = -K_{ba}$, we have $b_{\max} = 1$,

$$K_{ab} = \overbrace{2K_{0i}\, n_{[a} m^i{}_{b]}}^{1} + \overbrace{2K_{01}\, n_{[a} \ell_{b]} + K_{ij}\, m^i{}_{[a} m^j{}_{b]}}^{0} + \overbrace{2K_{1i}\, \ell_{[a} m^i{}_{b]}}^{-1}.$$

If $K_{0i} = 0$ then ℓ is aligned. If in addition, $K_{01} = K_{ij} = 0$, then the multiplicity of ℓ is 2.
In general types G, II, D, N (in even dimensions: type II or more special, in 4D: D, N)

$$C_{abcd} = C_{\{abcd\}} \equiv \frac{1}{2}(C_{[ab][cd]} + C_{[cd][ab]}), \quad C^{c}{}_{acb} = 0, \quad C_{a[bcd]} = 0.$$

$$\boldsymbol{C} = (\boldsymbol{C})_{(+2)} + (\boldsymbol{C})_{(+1)} + (\boldsymbol{C})_{(0)} + (\boldsymbol{C})_{(-1)} + (\boldsymbol{C})_{(-2)}.$$

$$\begin{aligned}
C_{abcd} = \; & \overbrace{4C_{0i0j}\, n_{\{a}m^{(i)}_{b}n_{c}m^{(j)}_{d\}}}^{\text{boost weight 2 — type G}} \\
& \overbrace{+8C_{010i}\, n_{\{a}l_{b}n_{c}m^{(i)}_{d\}} + 4C_{0ijk}\, n_{\{a}m^{(i)}_{b}m^{(j)}_{c}m^{(k)}_{d\}}}^{\text{1, I}} \\
& \left.\begin{aligned} &+4C_{0101}\, n_{\{a}l_{b}n_{c}l_{d\}} + 4C_{01ij}\, n_{\{a}l_{b}m^{(i)}_{c}m^{(j)}_{d\}} \\ &+8C_{0i1j}\, n_{\{a}m^{(i)}_{b}l_{c}m^{(j)}_{d\}} + C_{ijkl}\, m^{(i)}_{\{a}m^{(j)}_{b}m^{(k)}_{c}m^{(l)}_{d\}} \end{aligned}\right\}\text{0, II,D} \\
& \overbrace{+8C_{101i}\, l_{\{a}n_{b}l_{c}m^{(i)}_{d\}} + 4C_{1ijk}\, l_{\{a}m^{(i)}_{b}m^{(j)}_{c}m^{(k)}_{d\}}}^{\text{−1, III}} \\
& \overbrace{+4C_{1i1j}\, l_{\{a}m^{(i)}_{b}l_{c}m^{(j)}_{d\}}}^{\text{−2, N}}\,.
\end{aligned}$$

In particular, **for type N** the Weyl tensor in an appropriate null frame

$$C_{abcd} = 4C_{1i1j}\,\ell_{\{a}m^{(i)}_{b}\ell_{c}m^{(j)}_{d\,\}}, \text{ where } T_{\{abcd\}} \equiv \frac{1}{2}(T_{[ab][cd]}+T_{[cd][ab]}).$$

Equivalently, for type N the null vector $\boldsymbol{\ell}$ obeys [Ortaggio, CQG 2009]

$$C_{ab[cd}\ell_{e]} = 0.$$

Note that for $d > 4$ this is **not** equivalent to the standard 4D Bel-Debever condition $C_{abcd}\ell^{d} = 0$.

Bel-Debever criteria

[Ortaggio 2009]

$\boldsymbol{\ell}$ is a WAND (type I) $\Leftrightarrow \ell_{[e} C_{a]bc[d} \ell_{f]} \ell^b \ell^c = 0,$

$\boldsymbol{\ell}$ is a WAND of multiplicity ≥ 2 (type II) $\Leftrightarrow \ell_{[e} C_{a]b[cd} \ell_{f]} \ell^b = 0,$

$\boldsymbol{\ell}$ is a WAND of mult. ≥ 3 (type III) $\Leftrightarrow \ell_{[e} C_{ab][cd} \ell_{f]} = 0 = C_{abc[d} \ell_{e]} \ell^c,$

$\boldsymbol{\ell}$ is a WAND of multiplicity 4 (type N) $\Leftrightarrow C_{ab[cd} \ell_{e]} = 0.$

Spin coefficients

Ricci rotation coefficients L_{ab}, N_{ab} and $\overset{i}{M}_{ab}$ are defined by

$$\ell_{a;b} = L_{cd}\, m_a^{(c)} m_b^{(d)} \;,\; n_{a;b} = N_{cd}\, m_a^{(c)} m_b^{(d)} \;,\; m_{a;b}^{(i)} = \overset{i}{M}_{cd}\; m_a^{(c)} m_b^{(d)} \;,$$

where $a, b = 0 \dots d-1$.

For example

$$\ell_{a;b} = L_{11}\ell_a\ell_b + L_{10}\ell_a n_b + L_{1i}\ell_a m_b^{(i)} + L_{i1} m_a^{(i)}\ell_b + L_{i0} m_a^{(i)} n_b + L_{ij} m_a^{(i)} m_b^{(j)} .$$

$\rho_{ij} \equiv L_{ij}$ - optical matrix

Let us decompose optical matrix ρ_{ij} into its tracefree symmetric part σ_{ij} (*shear*), its trace θ (*expansion*) and its antisymmetric part A_{ij} (*twist*)

$$\rho_{ij} = \sigma_{ij} + \theta\delta_{ij} + A_{ij},$$
$$\sigma_{ij} \equiv L_{(ij)} - \tfrac{1}{d-2} L_{kk}\delta_{ij}, \qquad \theta \equiv \tfrac{1}{d-2} L_{kk}, \qquad A_{ij} \equiv L_{[ij]}.$$

$\boldsymbol{\ell}$ is geodetic iff $L_{i0} = 0$.
Optical scalars: other scalar quantities (apart from expansion) out of $\ell_{a;b}$: shear and twist

$$\sigma^2 \equiv \sigma_{ii}^2 = \sigma_{ij}\sigma_{ji}, \qquad \omega^2 \equiv -A_{ii}^2 = -A_{ij}A_{ji}.$$

If $\boldsymbol{\ell}$ is *affinely parametrized*, i.e. $L_{10} = 0$, the optical scalars take the form

$$\sigma^2 = \ell_{(a;b)}\ell^{(a;b)} - \tfrac{1}{d-2}\left(\ell^a_{\ ;a}\right)^2, \qquad \theta = \tfrac{1}{d-2}\ell^a_{\ ;a}, \qquad \omega^2 = \ell_{[a;b]}\ell^{a;b}.$$

Definition

Spacetimes admitting a null geodesic field $\boldsymbol{\ell}$ with the optical matrix of the form

- $\rho_{ij} = 0$ Kundt
- $\rho_{ij} \propto \delta_{ij}$ Robinson-Trautmann

Classification of Kundt and RT geometries

[Podolský, Ortaggio 2006], [Ortaggio, Pravda, Pravdová 2007], [Podolský, Švarc 2015]

Classification of Kundt and RT spacetimes, $d > 4$

General case

	RT	Kundt
Weyl type	I	I
Ricci type	G	I

Einstein spaces

	RT	Kundt
Weyl type	D	II

- Kundt: pp-waves are of Weyl type II and Ricci type II or more special (more restricted for $d = 4, 5$)
- RT: Schwarzschild-Tangherlini BH is of Weyl type D, Ricci type O
- RT: [Lu, Perkins, Pope, Stelle, 2015] BH in 4D quadratic gravity is of Weyl type D, Ricci type G (see [Pravda, Pravdová, Podolský, Švarc 2016])
- In fact all static, spherically symmetric spacetimes are type D RT.

It is of interest to study the behaviour of Kundt and RT metrics under conformal transformations.

[Pravda, Pravdová, Podolský, Švarc 2016]

All spacetimes conformal to Kundt are RT or Kundt. All RT spacetimes are conformal to Kundt.

Note that while conformal transformations preserve the Weyl type, they do not preserve the Ricci type.

Goldberg-Sachs theorem

So far we discussed two special forms of the optical matrix $\rho_{ij} = 0$ and $\rho_{ij} \propto \delta_{ij}$. What are other possible forms? This depends on the dimension, algebraic type and on the theory of gravity we study.

[Pravda, Pravdová, Coley, Milson, 2004]

For a type N Einstein spacetime, the optical matrix corresponding to the mWAND admits the form

$$\rho_{ij} = \begin{pmatrix} b & a & 0 & \dots \\ -a & b & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus there are no type N Einstein RT spacetimes for $d > 4$.
Indeed $d > 4$ Einstein RT spacetime are of type D [Podolský, Ortaggio 2006].
This is in contrast with 4D results.

However, for some theories type N Robinson-Trautman vacuum solutions do exist:

[Pravda, Pravdová, Podolský, Švarc 2016]

Type N Robinson-Trautman metrics conformal to type N universal Kundt spacetimes solve the vacuum equations of $d \geq 4$ conformal gravities.

4D Goldberg-Sachs theorem

In an Einstein spacetime which is not conformally flat, a null vector field is a repeated PND if, and only if, it is geodesic and shear-free.

Algebraically special $\Longleftrightarrow$ ℓ geodesic and $\rho_{ij} = b \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}$

Examples:

- $a \neq 0$: Kerr
- $a = 0$: Schwarzschild
- $a = 0 = b$: Kundt

Goldberg-Sachs theorem in HD

In contrast with the $d = 4$ case, in higher dimensions there exist vacuum spacetimes with non-geodetic multiple WANDs

[Pravda, Pravdová, Ortaggio, 2007], [Durkee, Reall, CQG, 2009].

It has been proven that

"Geodetic" part of the Goldberg-Sachs theorem [Durkee, Reall, CQG, 2009]

A vacuum spacetime admitting a *non-geodetic* multiple WAND always also admits another multiple *geodetic* WAND.

GS theorem in 5D [Ortaggio, Pravda, Pravdová, Reall, 2012]

In a 5d algebraically special Einstein spacetime that is not conformally flat, there exists a geodesic multiple WAND $\boldsymbol{\ell}$ and one can choose the orthonormal basis vectors $\boldsymbol{m}^{(i)}$ so that the optical matrix of $\boldsymbol{\ell}$ takes one of the forms

$$i) \quad b\begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1+a^2 \end{pmatrix},$$

$$ii) \quad b\begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$iii) \quad b\begin{pmatrix} 1 & a & 0 \\ -a & -a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If the spacetime is type III or type N then the form must be ii).

6 parametres $\rightarrow$ 2 parameters

case i): 5D Kerr-de Sitter (or some analytic continuation), type D

[de Freitas, Godazgar, Reall, 2015]

case iii): type D, non-geodesic multiple WAND [Wylleman 2015], [Durkee, Reall, 2009]

case ii) [Reall, Graham, Turner 2013], [de Freitas, Godazgar, Reall, 2016]

Algebraically special solutions of generalized gravities

For tensors $\mathbf{A}$, $\mathbf{B}$

$$\mathrm{bo}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{bo}(\mathbf{A}) + \mathrm{bo}(\mathbf{B})$$

while tensor contraction cannot increase the boost order.
Let $\mathrm{bo}(\mathbf{A}) = \mathrm{bo}(\mathbf{B}) = -2.$ Then any rank 2 tensor constructed from $\mathbf{A}$ and $\mathbf{B}$ vanishes.
In generalized theories of gravity, additional rank 2 terms of higher order in curvature and its derivatives appear in the field equations. These considerably simplify for Weyl types III and N.

For Weyl type N, all rank-2 tensors at least quadratic in Weyl vanish.
For Weyl type III, all rank-2 tensors at least cubic in Weyl vanish.

Quadratic gravity

$$S=\int \mathrm{d}^n x\sqrt{-g}\left(\frac{1}{\kappa}\left(R-2\Lambda_0\right)+\alpha R^2+\beta R_{ab}^2+\gamma\left(R_{abcd}^2-4R_{ab}^2+R^2\right)\right)$$

$\Rightarrow$ quadratic gravity field equations [Gullu, Tekin, Phys. Rev. D, 2009]

$$\frac{1}{\kappa}\left(R_{ab}-\frac{1}{2}Rg_{ab}+\Lambda_0 g_{ab}\right)+2\alpha R\left(R_{ab}-\frac{1}{4}Rg_{ab}\right)+(2\alpha+\beta)\left(g_{ab}\nabla^c\nabla_c-\nabla_a\nabla_b\right)R$$
$$+2\gamma\left(RR_{ab}-2R_{acbd}R^{cd}+R_{acde}R_b{}^{cde}-2R_{ac}R_b{}^c-\frac{1}{4}g_{ab}\left(R_{cdef}^2-4R_{cd}^2+R^2\right)\right)$$
$$+\beta\nabla^c\nabla_c\left(R_{ab}-\frac{1}{2}Rg_{ab}\right)+2\beta\left(R_{acbd}-\frac{1}{4}g_{ab}R_{cd}\right)R^{cd}=0.$$

For type N, Einstein all quadratic gravity terms are proportional to the metric. Such spacetimes are thus " immune" to the quadratic gravity corrections and are vacuum solutions to both Einstein gravity and quadratic gravity.

[T. Málek, V. P., 2011]

In arbitrary dimension, all Weyl type N Einstein spacetimes are vacuum solutions of quadratic gravity.

QG in four dimensions

QG has special properties in 4D - all Einstein spacetime are its vacuum solutions. Non-trivial solutions can be constructed by an appropriate conformal transformation of Einstein or Bach-free spacetimes [Pravda, Pravdová, Podolský, Švarc 2016].
Kundt backgrounds $\Rightarrow$Kundt/RT solutions. RT are of generic Ricci type, while Weyl is usually algebraically special.

In **more general theories of gravity**, covariant derivatives of the Weyl tensor also appear in the field eqs., e.g.
$C^{pqrs}C_{pqrs;ab}, \quad C^{pqrs}{}_{;a}C_{pqrs;b}, \quad C^{pqr}{}_{a;s}C_{pqrb}{}^{;s},$

$\mathrm{bo}(\mathrm{C})$=-2 $\Rightarrow$ $\mathrm{bo}(\nabla \mathbf{C}) = -2$ **???**
This is **not true in general**.

Using higher-dimensional NP/GHP formalism one can prove:

Proposition [Hervik, Pravda, Pravdová, 2014]

i) For type N Einstein Kundt spacetimes, the boost order of $\nabla^{(k)} C$ with respect to the multiple WAND is at most -2.
ii) For these spacetimes, all rank-2 tensors constructed from the Riemann (Weyl) tensor and its derivatives of arbitrary order are proportional to the metric (and in most cases in fact vanish).

Since for a type N Einstein spacetime, one cannot construct a non-trivial rank-2 tensor from the curvature and its covariant derivatives, it follows that such a spacetime will obey vacuum field equations of any theory with the Lagrangian of the form

$$L = L(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \dots, \nabla_{a_1 \dots a_p} R_{bcde})$$

with L being a polynomial curvature invariant.

Note that type III and II universal spacetimes also exist.

Conclusions

- Tensors on d-dimensional Lorentzian manifolds can be invariantly classified to classes G, I, II, D, III, N.
- Many known exact solutions to HD Einstein gravity or to generalized gravities have algebraically special Weyl tensor.
- Two important classes of solutions are Kundt and Robinson-Trautman. Robinson-Trautman spacetimes are more restricted by the Einstein equations for $d > 4$. However, this result does not extend to (some) generalized gravities (e.g. for HD conformal gravities, type N/III RT solutions do exist - spacetimes conformal to type N/III universal Kundt spacetimes)

Conclusions

- For type N and III spacetimes, the field equations of generalized gravities are considerably simplified. For example, for type N Einstein all quadratic and higher order terms in curvature are trivial.
- For certain Kundt spacetimes, this can be extended also to terms constructed from covariant derivatives of curvature. This leads to universal spacetimes - vacuum solutions to all theories of gravity with
  $L = L(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{a_1 \ldots a_p} R_{bcde})$.