Universal Walker metrics

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Outline

1. Motivation for universal metrics

2. Universal metrics

3. Neutral signature

4. Walker metrics

Quadratic gravity

- correction terms quadratic in the Riemann tensor

$$\mathcal{L} = \frac{1}{\kappa}\left(R - 2\Lambda_0\right) + \alpha R^2 + \beta R_{ab}^2 + \gamma\left(R_{abcd}^2 - 4R_{ab}^2 + R^2\right)$$

- field equations of fourth order

$$\frac{1}{\kappa}\left(R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda_0 g_{ab}\right) + 2\alpha R\left(R_{ab} - \frac{1}{4}Rg_{ab}\right) + (2\alpha + \beta)\left(g_{ab}\Box - \nabla_a\nabla_b\right)R$$

$$+ 2\gamma\left(RR_{ab} - 2R_{acbd}R^{cd} + R_{acde}R_b{}^{cde} - 2R_{ac}R_b{}^c - \frac{1}{4}g_{ab}\left(R_{cdef}^2 - 4R_{cd}^2 + R^2\right)\right)$$

$$+ \beta\Box\left(R_{ab} - \frac{1}{2}Rg_{ab}\right) + 2\beta\left(R_{acbd} - \frac{1}{4}g_{ab}R_{cd}\right)R^{cd} = 0$$

Solutions immune to the corrections

- corrections can be considered as effective energy-momentum tensor

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda_0 g_{ab} = \kappa T_{ab}^{\text{eff}}$$

- looking for solutions with $T_{ab}^{\text{eff}} = 0$
  or more general $T_{ab}^{\text{eff}} \propto g_{ab}$
- exact **vacuum solutions to both** Einstein gravity and QG

T.M., Pravda (2011)

- all Weyl type N Einstein metrics are exact vacuum solutions of QG
- for Weyl type III case, additional condition $C_a{}^{cde}C_{bcde} = 0$ has to be met

Universal metrics

A metric is **universal** if **all conserved symmetric rank-2 tensors** constructed from the metric, the Riemann tensor and its covariant derivatives of all orders are multiples of the metric.

- **universal metrics** solve **all theories** with the Lagrangian being a polynomial curvature invariant

$$\mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \dots, \nabla_{a_1 \dots a_p} R_{bcde})$$

- universal metrics are necessarily Einstein

$$R_{ab} = \frac{R}{n} g_{ab}, \quad R = \text{const}$$

and thus

$$R_{abcd} = C_{abcd} + \frac{2R}{n(n-1)} g_{a[c} g_{d]b}, \quad R_{abcd;e} = C_{abcd;e}$$

Examples of universal metrics

Hervik, Pravda, Pravdová (2014)

- Weyl type N ST is universal iff it is an Einstein Kundt ST
- Weyl type III Einstein RNV ST obeying $C_a{}^{cde}C_{bcde} = 0$ are universal

Hervik, T.M., Pravda, Pravdová (2015)

- direct product of non-flat maximally symmetric spaces M_α

$$M = M_0 \times M_1 \times \cdots \times M_{N-1}$$

M is Weyl type D universal iff $n_i = n_j$ and $R_i = R_j$

- M is Weyl type II universal if moreover M_0 is a type III or N universal spacetime

- all known examples of universal metrics belong to the **Kundt** class

Four-dimensional metrics of neutral signature

- purely real null frame $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \tilde{\boldsymbol{m}}\}$

$$\ell_a n^a = 1, \quad m_a \tilde{m}^a = -1$$

  all other contractions vanish

- 2 independent boosts

$$(\boldsymbol{\ell}, \boldsymbol{n}) \mapsto (\mathrm{e}^{\lambda_1} \boldsymbol{\ell}, \mathrm{e}^{-\lambda_1} \boldsymbol{n})$$
$$(\boldsymbol{m}, \tilde{\boldsymbol{m}}) \mapsto (\mathrm{e}^{\lambda_2} \boldsymbol{m}, \mathrm{e}^{-\lambda_2} \tilde{\boldsymbol{m}})$$

- boost weight $\mathbf{b} \equiv (b_1, b_2)$ of a quantity q

$$q \mapsto \mathrm{e}^{b_1 \lambda_1 + b_2 \lambda_2} q$$

Spin coefficients

- neutral signature analogue of the NP formalism [P. R. Law (2009)]
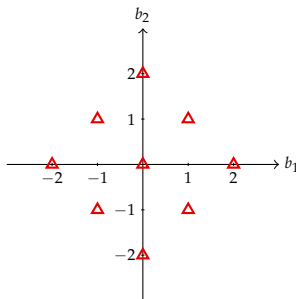- 24 real functions

- Kundt metric $\Leftrightarrow \kappa = \tilde{\kappa} = \sigma = \tilde{\sigma} = \rho = \tilde{\rho} = 0$
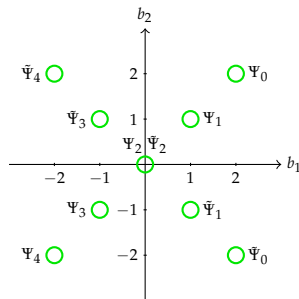 moreover for pp-frame $\varepsilon = \tilde{\varepsilon} = 0$

Algebraic classification

- boost-weight decomposition of a tensor $\mathbf{T}$

$$\mathbf{T} = \sum_{\mathbf{b} \in \mathbb{Z}^2} (\mathbf{T})_{\mathbf{b}}$$

rank-2 tensor

Weyl tensor

Walker metrics

Walker metrics in dimension n admit an invariant null k-plane with $k \leq n/2$
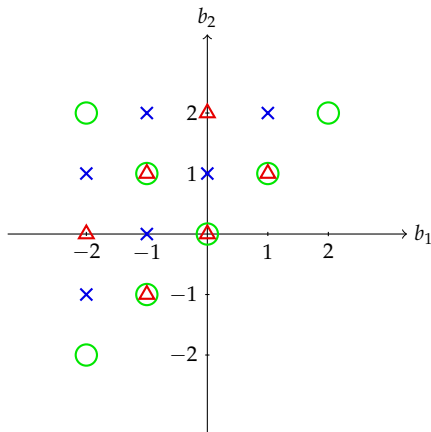
- canonical form for $n = 4$, $k = 2$

$$ds^2 = 2du(dv + a\,du + c\,dU) + 2dU(dV + b\,dU)$$

- invariant null 2-planes are spanned by $\boldsymbol{\ell} \equiv \partial_v$, $\tilde{\boldsymbol{m}} \equiv \partial_V$
- null plane – mutually orthogonal null vectors $\rightarrow$ neutral signature
- invariant plane

$$\nabla_a(\boldsymbol{\ell} \wedge \tilde{\boldsymbol{m}}) = k_a(\boldsymbol{\ell} \wedge \tilde{\boldsymbol{m}})$$
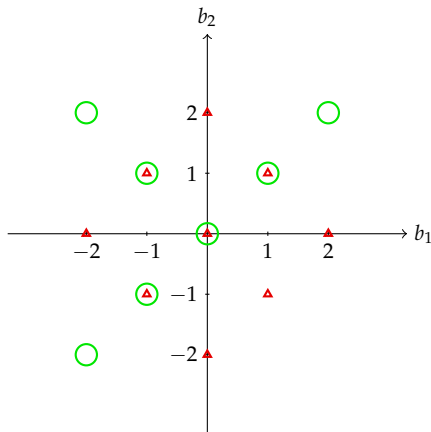
Boost-weight diagram of the general Walker metric

($\times$ spin coefficients, $\triangle$ trace-free Ricci tensor, $\bigcirc$ Weyl tensor)
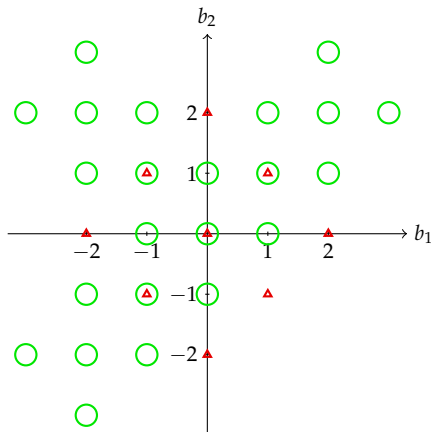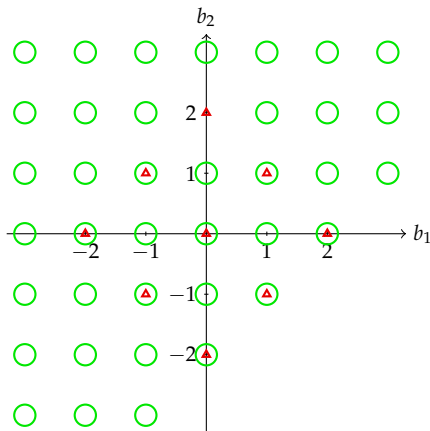
Boost-weight diagram of the general Walker metric

($\triangle$ rank-2 tensor, $\bigcirc$ $\nabla^{(k)}\mathbf{C}$ with $k = 0$)

Boost-weight diagram of the general Walker metric

($\triangle$ rank-2 tensor, $\bigcirc$ $\nabla^{(k)}\mathbf{C}$ with $k = 0, 1$)
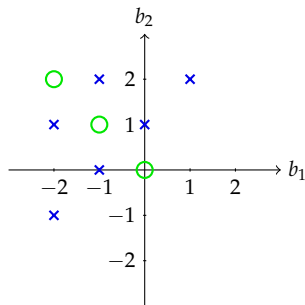
Boost-weight diagram of the general Walker metric

($\triangle$ rank-2 tensor, $\bigcirc$ $\nabla^{(k)}\mathbf{C}$ with $k = 0, 1, 2$)

Einstein Walker metrics with ASD Weyl tensor

$$\mathbf{C} = \mathbf{C}^{(-2,2)} + \mathbf{C}^{(-1,1)} + \mathbf{C}^{(0,0)}$$

$$a = \frac{\lambda}{3}v^2 + a_{10}\,v + a_{01}V + a_{00},$$

$$b = \frac{\lambda}{3}V^2 + b_{10}\,v + b_{01}V + b_{00},$$

$$c = \frac{2\lambda}{3}vV + c_{10}\,v + c_{01}V + c_{00},$$

- $\nabla^{(k)}\mathbf{C}^{(-2,2)}$ with $k \geq 0$ do not contribute to rank-2 tensors
  $\nabla^{(k)}\mathbf{C}^{(-1,1)}$ with $k \geq 0$ and $\nabla^{(k)}\mathbf{C}^{(0,0)}$ with $k > 0$ only linear
- possible contributions to rank-2 tensors ($k \geq 0$, $m \geq 0$)

$$\nabla^{(k)}\mathbf{C}^{(0,0)}\underbrace{\mathbf{C}^{(0,0)}\cdots\mathbf{C}^{(0,0)}}_{m \text{ times}}, \qquad \nabla^{(k)}\mathbf{C}^{(-1,1)}\underbrace{\mathbf{C}^{(0,0)}\cdots\mathbf{C}^{(0,0)}}_{m \text{ times}}$$

Universal Walker metrics [Hervik, T.M. (soon on arXiv)]

Einstein Walker metrics with anti-self-dual Weyl tensor are **0-universal**

- conserved symmetric rank-2 tensor constructed from the metric and the Riemann tensor (no ∇R_{abcd}) are multiples of the metric
- field equations may contain ∇R_{ab} (e.g. quadratic gravity)

Ricci-flat Walker metrics with anti-self-dual Weyl tensor are **universal**

- $\mathbf{C}^{(0,0)} = 0$ ($\tilde{\Psi}_2 = 0$) for Ricci-flat Walker metrics

Einstein Walker metrics with anti-self-dual Weyl tensor and $\tilde{\Psi}_3 = 0$ are **universal**

- if $\mathbf{C}^{(-1,1)} = 0$ ($\tilde{\Psi}_3 = 0$), $\nabla^{(k)}\mathbf{C}^{(0,0)}$ with $k > 0$ do not contribute and rank-2 tensors $\mathbf{C}^{(0,0)} \cdots \mathbf{C}^{(0,0)}$ are proportional to the metric