

Future Stability of the FLRW solutions $\Lambda > 0$

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Einstein - Euler equations

$$\left. \begin{aligned} \tilde{G}^{ij} &= \tilde{T}^{ij} - \Lambda \tilde{g}^{ij} \quad (\Lambda > 0) \\ \tilde{\nabla}_i \tilde{T}^{ij} &= 0 \end{aligned} \right\} \text{ on } \tilde{M} = [0, \infty) \times \mathbb{T}^3$$

where

$$\tilde{g} = \tilde{g}_{ij} dx^i dx^j$$

Indexing: $i, j, k \quad 0, 1, 2, 3$
 $I, \bar{J}, K \quad 1, 2, 3$

$$\tilde{T}^{ij} = (\rho + p) \tilde{v}^i \tilde{v}^j + p \tilde{g}^{ij} \quad ; \quad p = K\rho \quad 0 < K \leq \frac{1}{3}$$

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FLRW Solutions

$$\tilde{g} = -dx^0 dx^0 + \tilde{a}(x^0)^2 \int_{\mathbb{T}^3} dx^I dx^J, \quad \tilde{v} = \partial_0, \quad \tilde{\rho}(x^0) = \frac{\rho_0}{(\tilde{a}(x^0))^{3(1+K)}}$$

$$\frac{d\tilde{a}}{dx^0} = \tilde{a} \sqrt{\frac{\Lambda}{3} + \frac{\tilde{\rho}}{3}}, \quad \tilde{a}(0) = 1$$

$\tilde{a}(x^0) \sim e^{\sqrt{\frac{\Lambda}{3}} x^0} \text{ as } x^0 \rightarrow \infty$

Previous Results

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$\Lambda > 0$

Wave ganges (1985)

Ringström's Method

2008 H. Ringström Einstein-Schrödinger

2013 H. Ringström Einstein-Vlasov

Stability of FLRW

2013 I. Rodnianski & J. Speck $0 < K < \frac{1}{3}$, irrot
2012 J. Speck $0 < K < \frac{1}{3}$
2015 J. Speck & M. Hadžić $K = 0$

Conformal Method

1986 H. Friedrich Vacuum

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Gauge source + Conformal transformation

Stability of FLRW $0 < K \leq \frac{1}{3}$ ($K=0$ is also ok.)

Conformal Einstein Euler equations

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Conformal metric

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Space time

$$M = (0, 1] \times \mathbb{T}^3$$

△ x^i - periodic coordinates on \mathbb{T}^3

△ $t = x^0$ - time coordinate on $(0, 1]$

Time orientation

- △ The future lies in the direction of *decreasing* t .
- △ $v^0 > 0 \Rightarrow v^\mu$ is *future* oriented.
- △ $\Phi \nearrow \infty$ as $t \searrow 0 \Rightarrow$ future timelike infinity is located at $t=0$.

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where

$$X^k = g^{ij} \Gamma_{ij}^k$$

$$Y^k = -2\nabla^k \Phi + \underbrace{\xi^k}_{\text{gauge source}} = \frac{2}{t} \left(g^{k0} + \frac{\Lambda}{3} \delta_0^k \right)$$

Field Equations

$$G^{ij} = T^{ij} := e^{4\Phi} \tilde{T}^{ij} - e^{2\Phi} \Lambda g^{ij} + 2(\nabla^i \nabla^j \Phi - \nabla^i \Phi \nabla^j \Phi) - (2\Box g + |\nabla \Phi|_g^2) g^{ij}$$
$$\nabla_i \tilde{T}^{ij} = -6\tilde{T}^{ij} \nabla_i \Phi + g_{lm} \tilde{T}^{lm} g^{ij} \nabla_i \Phi$$

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Background metric

$$g^{\dot{ij}} = -\frac{\Lambda}{3} \delta_0^{\dot{i}} \delta_0^{\dot{j}} + \delta_I^{\dot{i}} \delta_J^{\dot{j}} \delta^{IJ}$$

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Auxiliary fields

$$g^{IJ} = \det(\check{g}_{LM})^{\frac{1}{3}} g^{IJ} \quad \left((\check{g}_{LM}) := (g^{LM})^{-1} \right)$$

$$g = g^{00} - \eta^{00} - \frac{\Lambda}{3} \ln(\det(g^{LM}))$$

Theorem 1.3. Suppose $\Lambda > 0$, $\epsilon > 0$, $0 < K \leq 1/3$, $k \in \mathbb{Z}_{\geq 3}$, $g_0^{ij} \in H^{k+1}(\mathbb{T}^3)$, $g_1^{ij}, \rho_0, v_I^0 \in H^k(\mathbb{T}^3)$, $\rho_0(x) > 0$ for all $x \in \mathbb{T}^3$, and that the quadruple

$$(g^{ij}, \partial_t g^{ij}, \rho, v_I)|_{t=1} = (g_0^{ij}, g_1^{ij}, \rho_0, v_I^0) \quad (1.14)$$

satisfies the constraint equations

$$(G^{i0} - T^{i0})|_{t=1} = 0, \quad \text{and} \quad Z^k|_{t=1} = 0. \quad (1.15)$$

Then there exists a $\delta > 0$, such that if

$$\|g_0^{ij} - \eta^{ij}\|_{H^{k+1}} + \|g_1^{ij}\|_{H^k} + \|\rho_0\|_{H^k} + \|v_I\|_{H^k} < \delta, \quad (1.16)$$

then there exists a classical solution $g^{ij} \in C^2((0, 1] \times \mathbb{T}^3)$, $\rho, v_I \in C^1((0, 1] \times \mathbb{T}^3)$ to the conformal Einstein-Euler equations given by (1.7) and (1.8) that satisfies the initial conditions (1.14), the gauge condition $Z^k = 0$ in $(0, 1] \times \mathbb{T}^3$, the regularity conditions

$$g^{ij} \in C^0((0, 1], H^{k+1}(\mathbb{T}^3)) \cap C^0([0, 1], H^k(\mathbb{T}^3)) \cap C^1((0, 1], H^k(\mathbb{T}^3)) \cap C^1([0, 1], H^{k-1}(\mathbb{T}^3))$$

and

$$\rho, v_I \in C^0((0, 1], H^k(\mathbb{T}^3)) \cap C^0([0, 1], H^{k-1}(\mathbb{T}^3)),$$

and the bounds

$$\frac{\Lambda}{6} \leq -g^{00}(t, x) \leq \frac{2\Lambda}{3}, \quad \frac{3}{2\Lambda} \leq -g_{00}(t, x) \leq \frac{6}{\Lambda}, \quad \frac{1}{2}\delta^{IJ} \leq g^{IJ}(t, x) \leq \frac{3}{2}\delta^{IJ},$$

and

$$\sqrt{\frac{3}{2\Lambda}} \leq v_0(t, x) \leq \sqrt{\frac{6}{\Lambda}}$$

for all $(t, x) \in (0, 1] \times \mathbb{T}^3$.

Moreover, there exist $\sigma, \gamma^j \in H^{k-1}(\mathbb{T}^3)$ with $\sigma(x) > 0$ for all $x \in \mathbb{T}^3$, such that the solution satisfies

$$\|g^{0j}(t) - (\eta^{0j} + t\gamma^j)\|_{H^{k-1}} \lesssim -t^2 \ln(t), \quad (1.17)$$

$$\|\partial_t g^{0j}(t) - 2t^{-1}(g^{0j}(t) - \eta^{0j}) + \gamma^j\|_{H^{k-1}} \lesssim t, \quad (1.18)$$

$$\|\partial_t g^{0j}(t) - t^{-1}(g^{0j}(t) - \eta^{0j})\|_{H^{k-1}} + \|\partial_I g^{0j}(t)\|_{H^{k-1}} \lesssim -t \ln(t), \quad (1.19)$$

$$\|\mathbf{q}(t) - \mathbf{q}(0)\|_{H^k} + \|\partial_t \mathbf{q}(t)\|_{H^{k-1}} \lesssim t, \quad (1.20)$$

$$\|\mathbf{g}^{IJ}(t) - \mathbf{g}^{IJ}(0)\|_{H^k} + \|\partial_t \mathbf{g}^{IJ}(t)\|_{H^{k-1}} \lesssim t, \quad (1.21)$$

$$\|t^{-3(1+K)}\rho(t) - \sigma\|_{H^{k-1}} \lesssim t + t^{\frac{2(1-3K)}{(1+\epsilon)}}, \quad (1.22)$$

and

$$\|v_I(t) - v_I(0)\|_{H^{k-1}} \lesssim t^{\frac{1-3K}{(1+\epsilon)}} \quad (1.23)$$

for all $t \in [0, 1]$, where $v_I(0) = 0$ if $K \neq 1/3$.

We conclude with a few remarks:

- (i) For any specified $\delta > 0$, an open set of initial data satisfying the constraint equations (1.15) and the condition (1.16) can be constructed using a variation of the method employed in [9, §3].
- (ii) The FLRW family of solutions correspond to the solutions obtained from initial data (1.14) of the form

$$(g_0^{ij}, g_1^{ij}, \rho_0, v_I^0) = (-a_0 \delta_0^i \delta_0^j + b_0 \delta_I^i \delta_J^j \delta^{IJ}, -a_1 \delta_0^i \delta_0^j + b_1 \delta_I^i \delta_J^j \delta^{IJ}, c, 0)$$

where the constants $a_0, b_0, c \in \mathbb{R}_{>0}$ and $a_1, b_1 \in \mathbb{R}$ are chosen so that the constraint equations (1.15) are satisfied.

- (iii) More detailed behaviour of the solution near $t = 0$ can be obtained by using the estimates (1.17)-(1.23) from Theorem 1.3 together with another application of Theorem B.1 from Appendix B.
- (iv) As discussed in Section 1.3 of [12], it should be possible using Ringström's patching argument from [11] to generalize the stability result contained in Theorem 1.3 to other spatial topologies.

Symmetric Hyperbolic Systems

$$B^i(t, x, u, v) \partial_i \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{t} \begin{pmatrix} B(t, x, u, v) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + H(t, x, u, v) \quad \text{in } [T_0, T_1) \times \mathbb{T}^n$$
$$(u, v)|_{t=0} = (u_0, v_0) \quad (T_1 < 0)$$

where

$$A \perp \leq B$$

and

$$B^0 = \begin{pmatrix} B_{11}^0 & B_{12}^0 \\ B_{21}^0 & B_{22}^0 \end{pmatrix}$$

satisfies

$$|B_{12}^0| + |B_{21}^0| \lesssim u$$

$$\left| DB_{11}^0 \cdot \left((B^0)^{-1} \begin{pmatrix} Bu \\ 0 \end{pmatrix} \right) \right| \lesssim t + 1$$

$$\left| DB_{12}^0 \cdot \left((B^0)^{-1} \begin{pmatrix} Bu \\ 0 \end{pmatrix} \right) \right| + \left| DB_{21}^0 \cdot \left((B^0)^{-1} \begin{pmatrix} Bu \\ 0 \end{pmatrix} \right) \right| \lesssim t + |u|$$

$$\left| DB_{22}^0 \cdot \left((B^0)^{-1} \begin{pmatrix} Bu \\ 0 \end{pmatrix} \right) \right| \lesssim t + |u|^2$$

Theorem B.1. Suppose $k \in \mathbb{Z}_{>n/2+1}$, $u_0 \in H^k(\mathbb{T}^n)$, $v, G \in C^0([T_0, 0], H^k(\mathbb{T}^n))$, assumptions (i)-(vii) are fulfilled, and the constants $\{\beta, \tilde{\kappa}, \tilde{\gamma}_1\}$ satisfy $0 \leq \beta/\tilde{\gamma}_1 < \tilde{\kappa}$ where $\beta = \sum_{i=1}^4 \beta_i$. Then there exists a $T_* \in (T_0, 0)$, and a unique classical solution $u \in C^1([T_0, T_*] \times \mathbb{T}^n)$ that satisfies $u \in C^0([T_0, T_*], H^k) \cap C^1([T_0, T_*], H^{k-1})$, the energy estimate

$$\|u(t)\|_{H^k}^2 + \|G\|_{L^\infty([T_0, 0], H^k)}^2 - \int_{T_0}^t \frac{1}{\tau} \|\mathbb{P}u(\tau)\|_{H^k}^2 \leq C e^{C(t-T_0)} \left(\|u(T_0)\|_{H^k}^2 + \|G\|_{L^\infty([T_0, 0], H^k)}^2 \right)$$

for $T_0 \leq t < T_*$ where

$$C = C(\|u\|_{L^\infty([T_0, T_*], H^k)}, \|v\|_{L^\infty([T_0, 0], H^k)}, \|G\|_{L^\infty([T_0, 0], H^k)}, \|\partial_t v\|_{L^\infty([T_0, 0], H^{k-1})}, \theta, \gamma_1, \gamma_2, \tilde{\kappa}, \beta, \omega),$$

and can be uniquely continued to a larger time interval $[T_0, T^*)$ for some $T^* \in (T_*, 0)$ provided that $\|u\|_{L^\infty([T_0, T_*], W^{1, \infty})} < \infty$.

Moreover, for any $R > 0$, there exists a

$$\delta = \delta(R, \|v\|_{L^\infty([T_0, 0], H^k)}, \|\partial_t v\|_{L^\infty([T_0, 0], H^{k-1})}, \theta, \gamma_1, \gamma_2, \beta, \tilde{\kappa}, \omega) > 0,$$

such that if $\|u_0\|_{H^k} + \|G\|_{L^\infty([T_0, 0], H^k)} \leq \delta$, then the solution $u(t, x)$ exists on the time interval $[T_0, 0]$ and can be uniquely extended to $[T_0, 0]$ as an element of $C^0([T_0, 0], H^{k-1})$ satisfying

$$\|u\|_{L^\infty([T_0, 0], W^{1, \infty})} \leq R,$$

and

$$\|\mathbb{P}u(t)\|_{H^{k-1}} \lesssim \begin{cases} -t & \text{if } \tilde{\kappa} - \beta_4/\tilde{\gamma}_1 > 1 \\ t \ln(\frac{t}{T_0}) & \text{if } \tilde{\kappa} - \beta_4/\tilde{\gamma}_1 = 1, \\ (-t)^{\tilde{\kappa} - \beta_4/\tilde{\gamma}_1} & \text{if } \tilde{\kappa} - \beta_4/\tilde{\gamma}_1 < 1 \end{cases},$$

$$\|\mathbb{P}^\perp u(t) - \mathbb{P}^\perp u(0)\|_{H^{k-1}} \lesssim \begin{cases} -t & \text{if } \tilde{\kappa} - \beta_4/\tilde{\gamma}_1 \geq 1 \text{ or } [B^0, \mathbb{P}] = 0 \\ -t + (-t)^{2(\tilde{\kappa} - \beta_4/\tilde{\gamma}_1)} & \text{if } \tilde{\kappa} - \beta_4/\tilde{\gamma}_1 < 1 \end{cases},$$

for $T_0 \leq t \leq 0$.

Future directions

- △ Understand when this method can be expected to work.
- △ Establish an instability result for $K > \frac{1}{3}$

A.D Rendall, *Asymptotics of solutions of the Einstein equations with positive cosmological constant*, Annales Henri Poincare **5** (2004), 1041-1064, [arXiv:gr-qc/0312020]

- △ Establish the existence of 1-parameter families of solutions that remain near Newtonian globally to the future.
(Work in progress with Chao Liu)