Characterization of (asymptotically) Kerr-de Sitter-like spacetimes at null infinity

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Kerr-(A)dS spacetime

One of the most important families of solutions to Einstein's Λ-vacuum field equations: Kerr-(Anti-)(de Sitter-)family

- depends on 3 parameters: Λ, m, a;
- two linearly independent Killing vectors;
- expected to satisfy black hole uniqueness results, and to
- describe the asymptotic state of large classes of evolution processes.

Aim: Characterization of Kerr-de Sitter (and related) spacetimes among spacetimes which

(i) solve the ($\Lambda > 0$)-vacuum equations,

(ii) admit a Killing vector field (KVF), and

(iii) admit a smooth conformal compactification à la Penrose,

in terms of asymptotic data at null infinity.

Conformally compactified spacetimes

A spacetime $(\mathscr{M}, g)$ has a smooth conformal compactification at infinity [Penrose '63, '65] supposing that

(i) there exists a spacetime $(\widetilde{\mathscr{M}}, \widetilde{g})$ and a conformal embedding ϕ,

$$\mathscr{M} \overset{\phi}{\hookrightarrow} \widetilde{\mathscr{M}} , \quad \phi^\star(\Theta^{-2}\widetilde{g}) = g \,, \quad \Theta \in C^\infty(\widetilde{\mathscr{M}}, \mathbb{R}) \,, \quad \Theta|_{\phi(\mathscr{M})} > 0 \,,$$

(ii) such that null infinity $\mathscr{I} := \partial\phi(\mathscr{M}) \cap \{\Theta = 0, \mathrm{d}\Theta \neq 0\}$ is a smooth hypersurface.

- If $(\widetilde{\mathscr{M}}, \widetilde{g})$ solves the $(\Lambda > 0)$-vacuum equations, $\mathscr{I}$ is a spacelike hypersurface.
- The Λ-vacuum equations are equivalent to a set of equations on $(\widetilde{\mathscr{M}}, \widetilde{g})$, the conformal field equations, which remain regular at $\{\Theta = 0\}$ [Friedrich '81].
- Weyl tensor $\widetilde{C}$ vanishes at $\mathscr{I} \Longrightarrow$ recaled Weyl tensor $\Theta^{-1}\widetilde{C}$ remains regular at $\mathscr{I}$.
- Permits construction of $\Lambda > 0$ vacuum spacetimes which admit a smooth conformal compactification at infinity in terms of an asymptotic Cauchy problem.

Asymptotic Cauchy problem and asymptotic Killing initial data sets

Theorem (Friedrich, 1986)

Let (Σ, h) be a Riemannian 3-manifold and D a symmetric 2-tensor. Then if and only if D is a TT-tensor, there exists a unique max. glob. hyp. development $(\widetilde{\mathscr{M}}, \widetilde{g})$ of the conformal $(\Lambda > 0)$-vacuum field equations such that

- $(\Sigma, h) = (\mathscr{I}^-,$ *induced metric on* $\mathscr{I}^-)$,
- $D = \lim_{\Theta \to \mathscr{I}^-} \Theta^{-1} \widetilde{C}(n, \cdot, n, \cdot)$ *(n unit future normal to $\mathscr{I}^-$).*

Theorem (P., 2014)

Let (Σ, h, D) be an asymptotic Cauchy data set for the conformal $(\Lambda > 0)$-vacuum eqns. The emerging vacuum spacetime $(\mathscr{M}, g = \Theta^{-2}\widetilde{g})$ admits a KVF X if and only if

(a) (Σ, h) *admits a conformal KVF (CKVF) Y,*

(b) *which satisfies the KID equation*

$$\mathscr{L}_Y D + \frac{1}{3}(\mathrm{div}_h Y) D = 0\,.$$

$\phi_\star(X)$ is tangential to $\mathscr{I}$ and its restriction to $\mathscr{I}$ can be identified with Y.

(Rescaled) Mars-Simon tensor

Definition

Let $(\mathscr{M}, g)$ be a Λ-vacuum space-time which admits a KVF X.

- self-dual Weyl tensor $\mathscr{C}_{\alpha\beta\mu\nu} := C_{\alpha\beta\mu\nu} + iC^{\star}_{\alpha\beta\mu\nu}$
- self-dual Killing form $\mathscr{F}_{\mu\nu} := \nabla_\mu X_\nu + i(\nabla_\mu X_\nu)^\star$, $\mathscr{F}^2 := \mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}$
- self-dual 4-tensor $\mathscr{I}_{\alpha\beta\mu\nu} := \frac{1}{4}(2g_{\alpha[\mu}g_{\nu]\beta} + i\epsilon_{\alpha\beta\mu\nu})$
- Mars-Simon tensor (MST): $\mathscr{S}_{\alpha\beta\mu\nu} := \mathscr{C}_{\alpha\beta\mu\nu} - Q\Big(\mathscr{F}_{\alpha\beta}\mathscr{F}_{\mu\nu} - \frac{1}{3}\mathscr{F}^2\mathscr{I}_{\alpha\beta\mu\nu}\Big)$, where $Q \in C^\infty(\mathscr{M}, \mathbb{C})$
- In $(\widetilde{\mathscr{M}}, \widetilde{g})$ we will deal with the rescaled MST $\mathscr{T} := \Theta^{-1}\mathscr{S}$.

Significance

- The Kerr-(A)dS spacetimes admit a distinguished KVF (for $m \neq 0$) for which the MST vanishes.
- In [Mars & Senovilla '15, '16] a complete classification of Λ-vacuum spacetimes is provided which admit a KVF whose associated MST vanishes for some function Q (in particular, a local characterization of the Kerr-(A)dS family).
- It was used to establish Kerr black hole uniqueness results without an analyticity assumption [Ionescu & Klainerman '09] [Alexakis, Ionescu & Klainerman '10].

Strategy to obtain characterization result

(A) Characterize asymptotic KIDs (Σ, h, D, Y) which yield a spacetime with an rescaled MST which vanishes on $\mathscr{I}$.

(B) Derive evolution equations for the rescaled MST which ensure that the MST vanishes everywhere.

(C) Classify the emerging spacetimes in terms of (Σ, h, D, Y).

Step (A)

Choice of the function Q:
Natural choice for $\mathscr{F}^2 \neq 0$:

$$\mathscr{S}_{\alpha\beta\mu\nu}\mathscr{F}^{\alpha\beta}\mathscr{F}^{\mu\nu} = 0 \quad \Longleftrightarrow \quad Q = Q_0 := \frac{3}{2}\mathscr{F}^{-4}\mathscr{C}_{\alpha\beta\mu\nu}\mathscr{F}^{\alpha\beta}\mathscr{F}^{\mu\nu}$$

Properties:

- $\mathscr{S} = 0 \Longrightarrow Q = Q_0$
- $\mathscr{S}^{(0)}$ vanishes at $\mathscr{I}$, whence the rescaled MST $\mathscr{T}^{(0)}$ is regular there.

Lemma (Mars, P., Senovilla & Simon, 2016)

Let (Σ, h, D, Y) by asymptotic KIDs with $|Y| > 0$. Then $\mathscr{T}^{(0)}|_{\mathscr{I}} = 0$ if and only if

$$\text{Cotton-York}(h) = A_{CY}|Y|^{-5}(Y \otimes Y)_{\mathrm{tf}}\,, \quad D = A_D|Y|^{-5}(Y \otimes Y)_{\mathrm{tf}}\,, \tag{1}$$

where $A_{CY}, A_D = \mathrm{const.}$

Problem: $\mathscr{T}^{(0)}$ does not seem to satisfy a useful evolution equation.

Step (B)

Alternative choice for Q:

Ernst potential σ is given, up to a complex "σ-constant", by $\nabla_\mu \sigma = 2X^\alpha \mathscr{F}_{\alpha\mu}$ (in Λ-vacuum the r.h.s. is closed). Set

$$Q_{\mathrm{ev}} := 3\sigma^{-1} + 4\Lambda \mathscr{F}^{-2} - 3\sigma^{-1}\sqrt{1 + 4\Lambda\sigma\mathscr{F}^{-2}}\ .$$

Properties:

- $\mathscr{S} = 0 \Longrightarrow Q = Q_{\mathrm{ev}}$ for some σ-constant [Mars & Senovilla '15].
- $\mathscr{T}^{(\mathrm{ev})}$ satisfies a homogeneous symmetric hyperbolic system of evolution equations which is of Fuchsian type at $\mathscr{I}$.
- Generically, $\mathscr{T}^{(\mathrm{ev})}$ will be singular at $\mathscr{I}$ – for any choice of the σ-constant.

Lemma (Mars, P., Senovilla & Simon, 2016)

Let (Σ, h, D, Y) by asymptotic KIDs. If and only if

$$\textit{Cotton-York}(h)(Y,\cdot) \propto Y\ , \quad D(Y,\cdot) \propto Y \tag{2}$$

there exists a choice of the σ-constant for which $\mathscr{T}^{(\mathrm{ev})}$ is regular at $\mathscr{I}$. In that case

$$\mathscr{T}^{(\mathrm{ev})}|_{\mathscr{I}} = \mathscr{T}^{(0)}|_{\mathscr{I}}\ .$$

Main result

Theorem (Mars, P., Senovilla & Simon, 2016)

Let (Σ, h, D, Y) by asymptotic KIDs with $|Y| > 0$. Then there exists a $\Lambda > 0$-vacuum spacetime $(\mathscr{M}, g)$ which admits a KVF X with $X^i|_{\mathscr{I}^-} = Y^i$, such that the associated MST vanishes, and $(\Sigma, h) = (\mathscr{I}^-, \text{induced metric on } \mathscr{I}^-)$ and $D = \lim_{\Theta \to \mathscr{I}^-} \Theta^{-1} \widetilde{C}(n, \cdot, n, \cdot)$ if and only if ($A_{CY}, A_D = \mathrm{const.}$)

$$\text{Cotton-York}(h) = A_{CY}|Y|^{-5}(Y \otimes Y)_{\mathrm{tf}}\,, \quad D = A_D|Y|^{-5}(Y \otimes Y)_{\mathrm{tf}}\,. \tag{1}$$

We call spacetimes generated by asymptotic KIDs which satisfy (2) asymptotically Kerr-de Sitter-like (note that (2) is weaker than (1)).

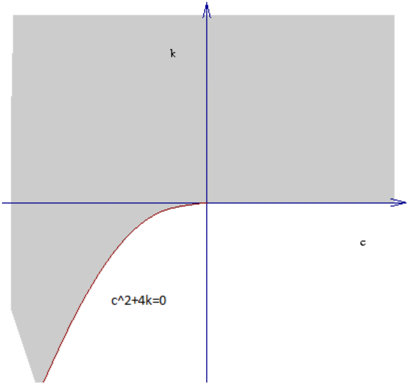
Step (C): Asymptotic characterization of KdS and related spacetimes

We make the additional assumption that $\mathscr{I}$ is conformally flat $\iff A_{CY} = 0$.

Only conformal class of $(\mathscr{I}, h)$ matters geometrically:

Take round sphere $(\mathbb{S}^3, s)$ as initial manifold and classify CKVFs up to conformal diffeomorphisms: For this, define two functions $c := c(Y)$, $k := k(Y)$:

- constant if (i) holds,
- depend only on $[Y]$.
- except for $\{k = 0, c < 0\}$ there exist at most one equivalence class of CKVF
- $\mathscr{I}$ can be identified with $\mathbb{S}^3 \setminus \{Y = 0\}$.



Theorem (Mars, P. & Senovilla, 2016)

Let (Σ, h, D, Y) by asymptotic KIDs. Then the emerging spacetime will be locally isometric to KdS if and only if
(a) (Σ, h) *is conformally flat,* *(b)* $D = A_D |Y|^{-5} (Y \otimes Y)_{\mathrm{tf}}$,
(c) $k(Y) > 0$ *or* $(k(Y) = 0$ *&* $c(Y) > 0)$.

Thank you!