The Abstract Boundary Singularity Theorem and its Generalisations

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Motivation

The end goal of our program of research is to link the Penrose Hawking Singularity Theorems to curvature singularity results.

Our three theorems presented here are a further step towards this goal.

They prove that the Penrose Hawking Singularity Theorems actually imply the existence of irremovable, also called essential, singularities.

They also provide a location for these singularities in terms of boundary points of an envelopment of a manifold.

It is hoped that this additional structure can be exploited to complete the program of research.

The Abstract Boundary

Let $\mathcal{C}$ be a family of parametrised curves in $\mathcal{M}$ such that

(i) for any point $p \in \mathcal{M}$ there is at least one curve γ of the family passing through p,

(ii) if γ is a curve of the family then so is every subcurve of γ, and

(iii) for any pair of curves γ and γ' in M which are obtained from each other by a change of parameter we have either that the parameter on both curves is bounded or it is unbounded on both curves.

An *enveloped manifold* is a triple $(\mathcal{M}\,\widehat{\mathcal{M}}, \phi)$ where $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are differentiable manifolds of the same dimension n and ϕ is a C^∞ embedding $\phi : \mathcal{M} \to \widehat{\mathcal{M}}$.

A *boundary point* p of an envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ is a point in the topological boundary of $\phi(\mathcal{M})$.

The Abstract Boundary

If B' is a boundary set of a second envelopment $(\mathcal{M}, \widehat{\mathcal{M}}', \phi')$ of $\mathcal{M}$ then we say B *covers* B' if for every open neighbourhood $\mathcal{U}$ of B in $\widehat{\mathcal{M}}$ there exists an open neighbourhood $\mathcal{U}'$ of B' in $\widehat{\mathcal{M}}'$ such that

$$\phi \circ \phi'^{-1} \left(\mathcal{U}' \cap \phi'(\mathcal{M}) \right) \subset \mathcal{U} \,. \tag{1}$$

Boundary sets B and B' are *equivalent* if B covers B' and B' covers B. This is an equivalence relation on the set of all boundary sets. An *abstract boundary set* is an equivalence class of boundary sets, denoted $[B]$.

For a manifold $\mathcal{M}$, an abstract boundary set is an *abstract boundary point* whenever it has a singleton $\{p\}$ as a representative boundary set. In this case the equivalence class is denoted by $[p]$. The set of all abstract boundary points is denoted $\mathcal{B}(\mathcal{M})$ and called the *abstract boundary* or *a-boundary* of $\mathcal{M}$.

Original Abstract Boundary Singularity Theorem

An abstract C^l essential singularity is an abstract boundary set which has a singleton $\{p\}$ as a representative boundary set where p is a singular boundary point which cannot be removed by a change of coordinates and is approached by a curve in $\mathcal{C}$ with bounded parameter.

Theorem 1 Ashley and Scott

Let $(\mathcal{M}\, g)$ be a strongly causal, C^l maximally extended, C^k spacetime $(1 \leq l \leq k)$. Let $\mathcal{C}$ be the set of affinely parametrised causal geodesics in $\mathcal{M}$. There exists an incomplete curve in $\mathcal{C}$ if and only if the Abstract Boundary $\mathcal{B}(\mathcal{M})$ contains an abstract C^l essential singularity.

This theorem does not prove the existence of an incomplete causal geodesic, but rather shows that the existence of an incomplete causal geodesic is equivalent to the existence of an endpoint for the incomplete geodesic—that is, a location for the singularity in the Abstract Boundary.

Causality Conditions and Causal Curves

Ideally, we want to relax the use of geodesics and the assumption of strong causality to increase the generality of the theorem. Geroch has shown that in order to identify all singular behaviour in a space-time it is necessary to consider, at least, all causal curves. Hence, it is desirable that we generalise this singularity theorem to include, at least, all causal curves.

The most general singularity theorems, like that given by Maeda and Ishibashi, use causality conditions much weaker than strong causality.

Ashley and Scott investigated weakening the strong causality condition in some cases. In particular, they have shown that the Abstract Boundary singularity theorem is false in chronological space-times and have indicated how a counter example may be provided in causal spacetimes.

Locally Lipschitz Curves

A (regular) locally Lipschitz curve $\gamma : [a, b) \to \mathcal{M}$ is a function so that, for each chart $\phi : U \subset \mathcal{M} \to \mathbb{R}^n$ and each $t \in [a, b)$ such that $\gamma(t) \in U$, there exists V, a neighbourhood of t in $[a, b)$, and $K \in \mathbb{R}^+$ such that $\gamma(V) \subset U$ and for all $t_1, t_2 \in V$,

$$d(\phi \circ \gamma(t_1), \phi \circ \gamma(t_2)) \leq K|t_1 - t_2|,$$

where d is the Euclidean distance on $\mathbb{R}^n$ and so that γ' is non-zero apart from a set of measure zero. Note that K depends on the chart ϕ and the point t.

The definition is independent of the choice of chart since the set V can be taken to be compact and changes of coordinates between charts are invertible and bounded on compact sets contained in the intersection of their domains.

Continuous Causal Curves

A continuous future directed, non-spacelike curve $\gamma : [a, b) \to \mathcal{M}$ is a continuous function so that for each $t_0 \in [a, b)$ there is a neighbourhood N of t_0 in $[a, b)$ and a convex normal neighbourhood U of $\gamma(t_0)$ so that for all $t \in N$, $t \neq t_0$, if $t > t_0$ then $\gamma(t) \in J^+(\gamma(t_0), U) - \gamma(t_0)$ or if $t < t_0$ then $\gamma(t) \in J^-(\gamma(t_0), U) - \gamma(t_0)$.

We say that γ is past directed if, for $t > t_0$, then $\gamma(t) \in J^-(\gamma(t_0), U) - \gamma(t_0)$ and for $t < t_0$ then $\gamma(t) \in J^+(\gamma(t_0), U) - \gamma(t_0)$.

We say that γ is timelike if the sets $I^+(\gamma(t_0), U)$ and $I^-(\gamma(t_0), U)$ are used instead of $J^+(\gamma(t_0), U) - \gamma(t_0)$ and $J^-(\gamma(t_0), U) - \gamma(t_0)$ respectively.

We assume that every continuous causal (timelike) curve is equipped with a parametrisation so that it is also a locally Lipschitz curve. That such a parametrisation always exists is proven in Beem, Ehrlich and Easley.

The Abstract Boundary Singularity Theorem

Defined a generalised affine parameter for locally Lipschitz curves in $\mathcal{M}$.

Theorem 2 *The Abstract Boundary Singularity Theorem*

Ashley, Scott, Whale

Let $(\mathcal{M}, g)$ be a future (past) distinguishing, C^l maximally extended, C^k spacetime $(1 \leq l \leq k)$ and let $\mathcal{C}$ be the family of generalised affinely parametrised continuous causal curves in $\mathcal{M}$. There exists an incomplete curve in $\mathcal{C}$ if and only if $\mathcal{B}(\mathcal{M})$ contains an abstract C^l essential singularity.

Theorem 3 Ashley, Scott, Whale

Let $(\mathcal{M}, g)$ be a C^l maximally extended, C^k spacetime $(1 \leq l \leq k)$ so that no inextendible locally Lipschitz curve is totally imprisoned. Let $\mathcal{C}$ be the family of generalised affinely parametrised locally Lipschitz curves in $\mathcal{M}$. There exists an incomplete curve in $\mathcal{C}$ if and only if $\mathcal{B}(\mathcal{M})$ contains an abstract C^l essential singularity.

Conclusions

Theorem 3 is more general than Theorem 2 but lacks physical motivation for the condition on $\mathcal{M}$.

Consider the three statements in Theorem 3.

(1) The space-time does not contain an inextendible totally imprisoned locally Lipschitz curve,

(2) The set of curves, $\mathcal{C}$, contains an incomplete curve,

(3) The Abstract Boundary contains an abstract C^l essential singularity.

(1) implies that (2) and (3) are equivalent

(2) implies that either (1) does not hold or (3) holds

(3) implies that (2) holds and that this incomplete curve is inextendible and not totally imprisoned

Conclusions

Hence, given the use of locally Lipschitz curves, Theorem 3 cannot be weakened further.

Theorem 3 can, therefore, be considered a proof of the generally accepted statement that incompleteness of curves implies either the existence of incomplete trapped curves or of a singularity.

B.E. Whale, M.J.S.L. Ashley and S.M. Scott "Generalizations of the abstract boundary singularity theorem" *Classical and Quantum Gravity* **32** (2015) 135001