Superradiant instabilities of asymptotically AdS black holes

Stephen R. Green

with Stefan Hollands, Akihiro Ishibashi and Robert M. Wald

21st International Conference on General Relativity and Gravitation
Columbia University
July 14, 2016

Based on: CQG **33**, 125022 (2016); 1512.02644 [gr-qc]

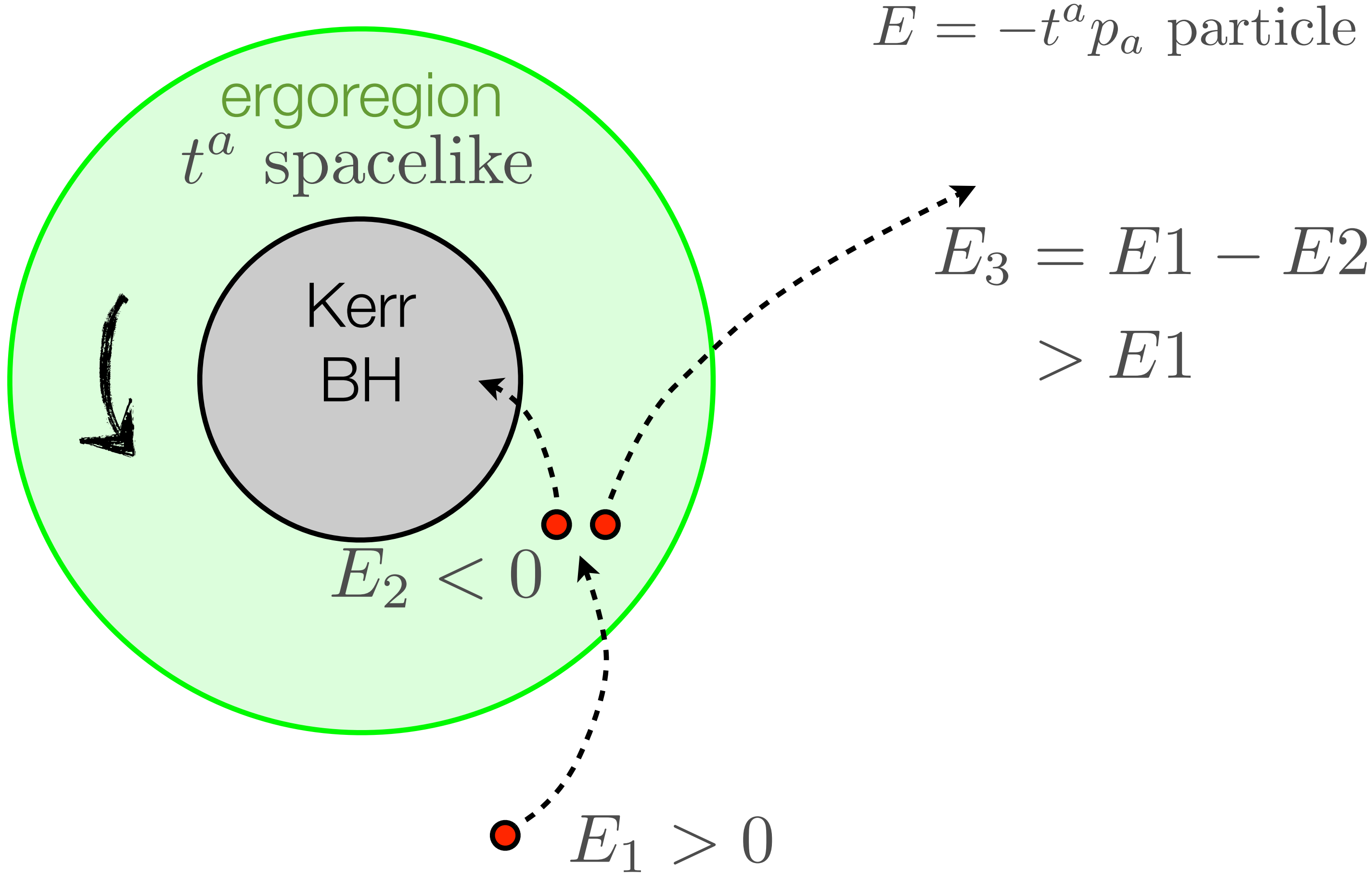
PERIMETER INSTITUTE
FOR THEORETICAL PHYSICS

Introduction to superradiant instability

- Mass and angular momentum can be extracted from a black hole with ergoregion.

  E.g., Penrose process

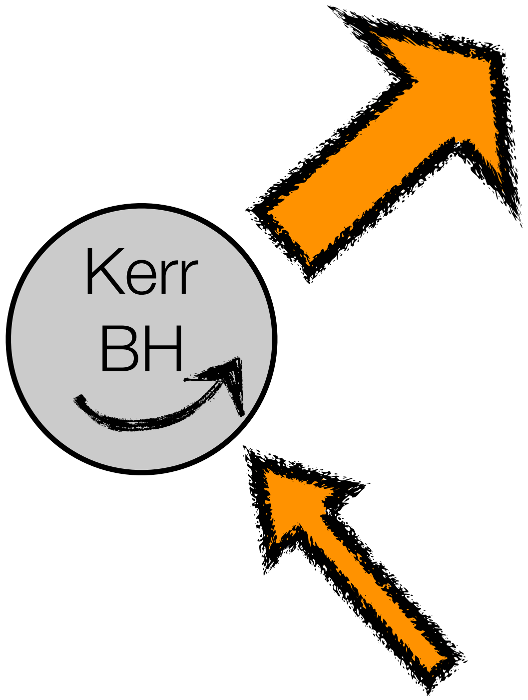
  t^a time translation Killing field
  $E = -t^a p_a$ particle energy



- Area law not violated since $A = 8\pi M\left[M + (M^2 - a^2)^{1/2}\right]$ and particles extract angular momentum as well.

Introduction to superradiant instability

- Similar process amplifies waves: superradiance

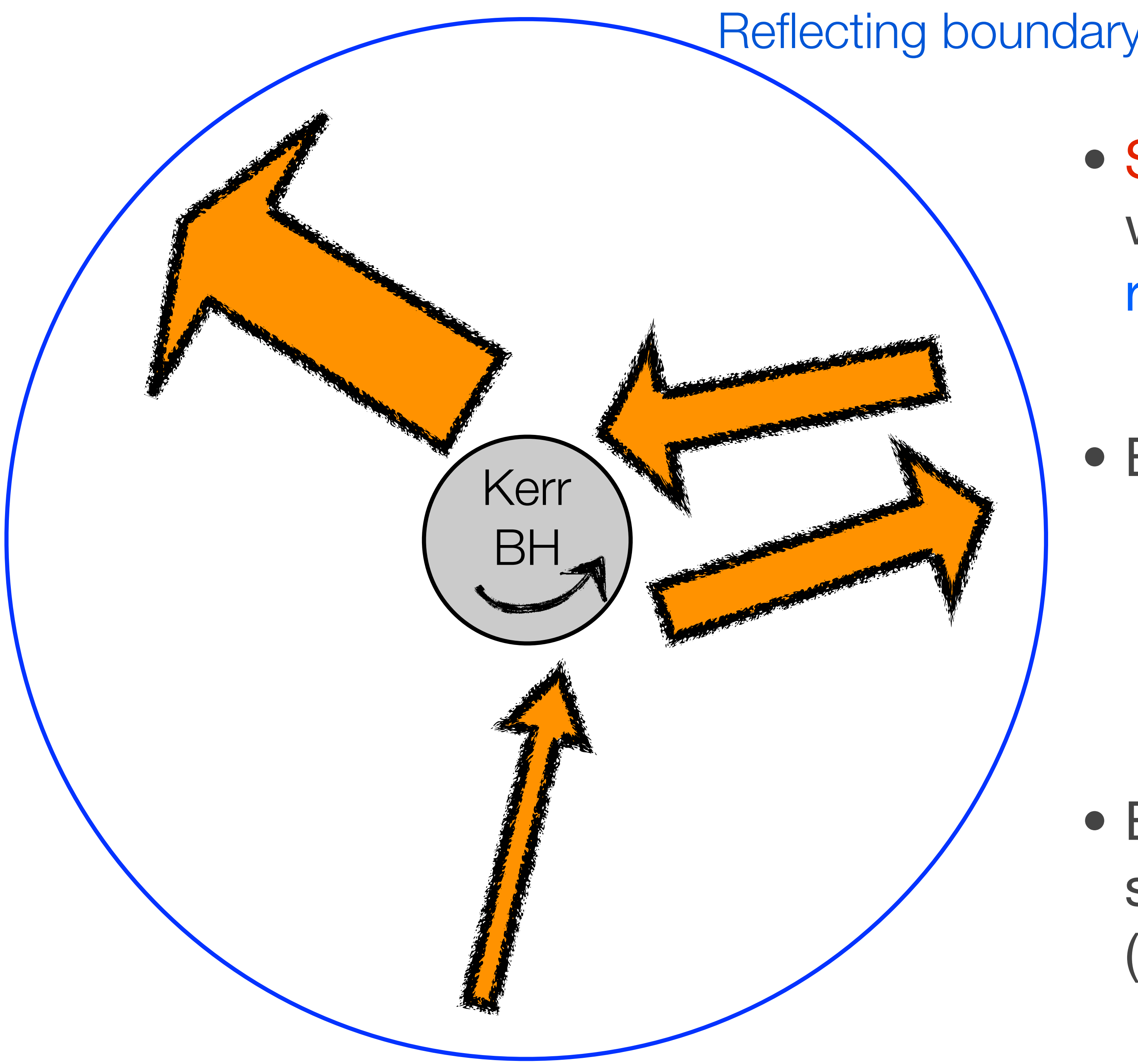


- Can be understood from the area theorem:
  - Wave $\sim e^{im\phi}e^{-i\omega t}$ changes BH area by

$$\frac{\kappa}{8\pi}\delta A = \delta M - \Omega_H \delta J$$
$$= \delta M \left(1 - \Omega_H \frac{\delta J}{\delta M}\right)$$
$$= \delta M \left(1 - \Omega_H \frac{m}{\omega}\right)$$

  - Thus, if $\boxed{0 < \omega < m\Omega_H}$, area increase requires $\delta M < 0$

Introduction to superradiant instability



- Superradiant instability caused when ergoregion combined with reflecting boundary.
- Examples:
  mass term for field
  mirror
  anti-de Sitter boundary
- Black hole must be sufficiently small, or else no ergoregion (e.g., Hawking-Reall bound)

Linear superradiant instability

- Background metric g_{ab}
  - asymptotically AdS black hole solution to Einstein equation in $d \geq 4$
  - horizon Killing vector field K^a
- Metric perturbation γ_{ab}
  - solution to linearized Einstein equation with reflecting AdS boundary condition
- Main result: Black hole is linearly unstable if K^a becomes spacelike somewhere outside the black hole (i.e., there is an ergoregion).

Canonical energy method

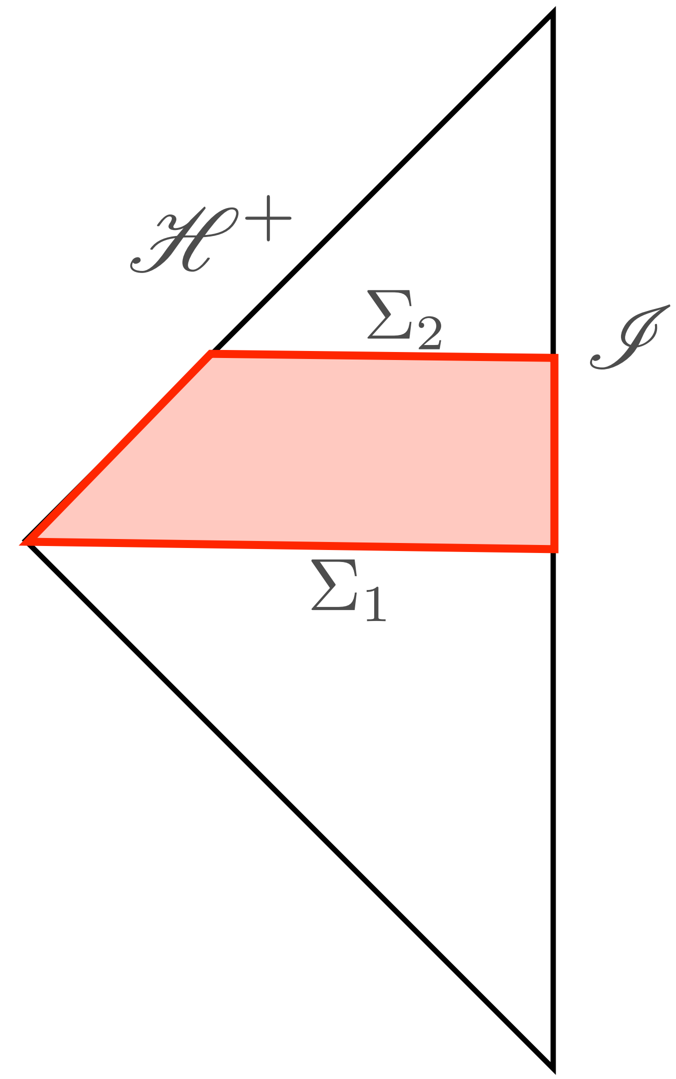
- Standard method to prove instability: Search for mode solutions that grow in time.

- This is difficult, in particular for complicated backgrounds, higher dimensions, or gravitational perturbations. Requires decoupling and separation of equations, which may not even be possible.

- Alternative is “canonical energy method”, which only requires construction of initial data solving the constraint equations---not a solution to the evolution equations.

Canonical energy method

- Canonical energy $\mathcal{E}$ is an integral over a Cauchy hypersurface Σ, quadratic in the perturbation γ_{ab}, satisfying
    - Gauge invariance
    - Degeneracy precisely on perturbations to other stationary black holes
    - Conservation
    - Positive flux at horizon and infinity



- Then $\mathcal{E}_{\Sigma_2} < \mathcal{E}_{\Sigma_1}$, and if a solution to the constraints γ_{ab} exists such that $\mathcal{E}_{\Sigma_1}(\gamma) < 0$, instability follows.

Construction of canonical energy

- Starting with Einstein-Hilbert action, derive symplectic current, which depends on two metric perturbations,

$$w^a(\gamma_1, \gamma_2) = \frac{1}{16\pi} g^{abcdef} \left(\gamma_{2bc} \nabla_d \gamma_{1ef} - \gamma_{1bc} \nabla_d \gamma_{2ef}\right),$$

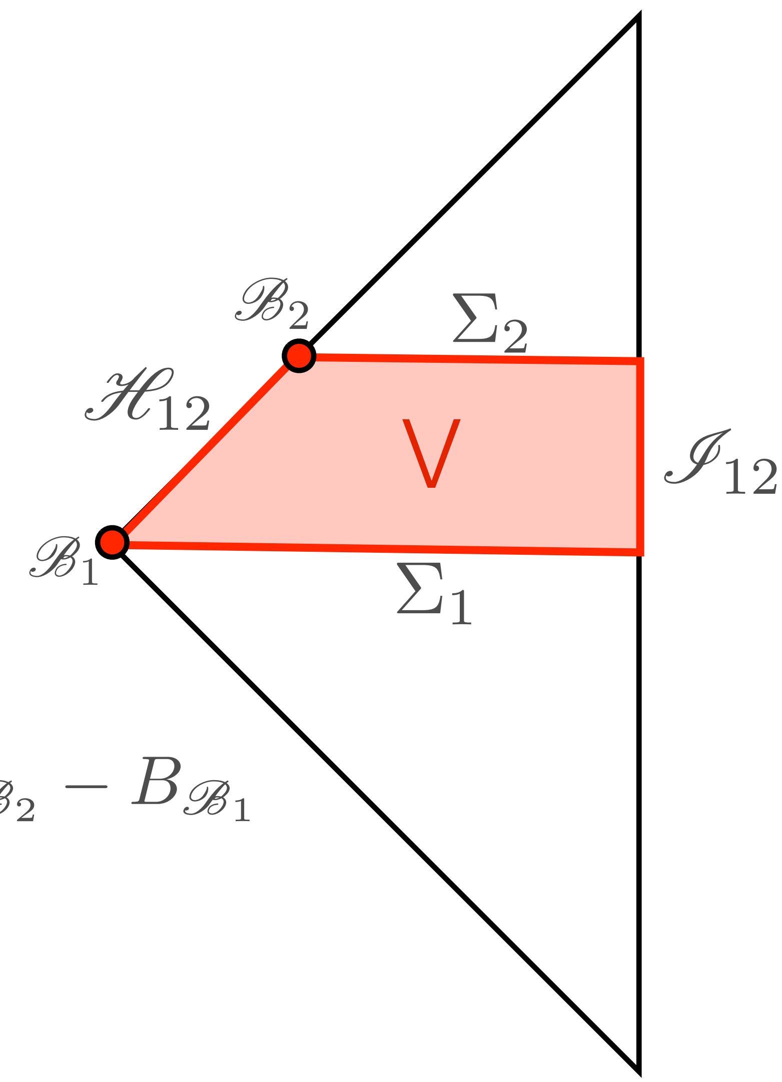
- For solutions to the linearized Einstein equation, $\boxed{\nabla_a w^a = 0}$

Positivity of fluxes

- Integrate over a volume V. On solutions, Stokes' theorem gives

$$0 = \int_V \nabla_a w^a = \int_{\partial V} n_a w^a$$

- Now take $\gamma_2 = \pounds_K \gamma_1$, so $w^a = w^a(\gamma, \pounds_K \gamma)$ and consider contributions from each boundary



$$\int_{\mathscr{I}_{12}} n_a w^a = 0$$

$$\int_{\mathscr{H}_{12}} n_a w^a = \frac{1}{4\pi} \int_{\mathscr{H}_{12}} (K^c \nabla_c u) \delta\sigma_{ab} \delta\sigma^{ab} + B_{\mathscr{B}_2} - B_{\mathscr{B}_1}$$

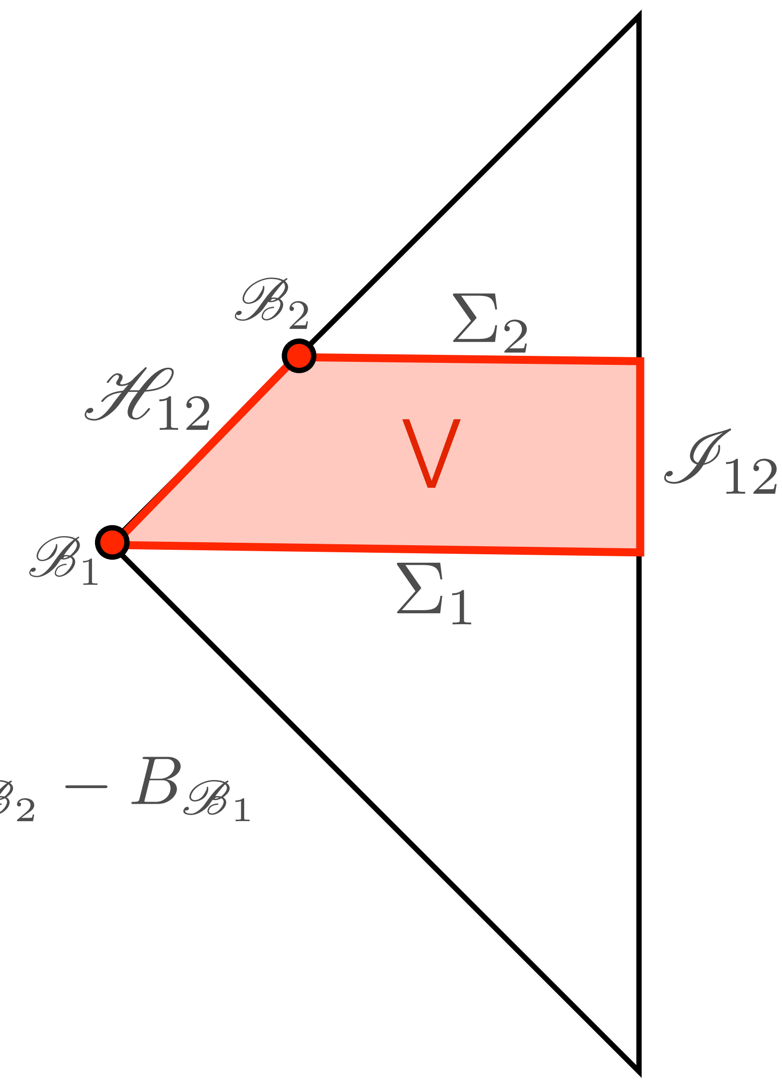
(imposed reflecting AdS boundary, and certain gauge conditions)

Positivity of fluxes

- Integrate over a volume V. On solutions, Stokes' theorem gives

$$0 = \int_V \nabla_a w^a = \int_{\partial V} n_a w^a$$

- Now take $\gamma_2 = \pounds_K \gamma_1$, so $w^a = w^a(\gamma, \pounds_K \gamma)$
and consider contributions from each boundary



$$\boxed{\int_{\mathscr{I}_{12}} n_a w^a = 0}$$

Flux through infinity vanishes in AdS

$$\int_{\mathscr{H}_{12}} n_a w^a = \frac{1}{4\pi} \int_{\mathscr{H}_{12}} (K^c \nabla_c u) \delta\sigma_{ab} \delta\sigma^{ab} + B_{\mathscr{B}_2} - B_{\mathscr{B}_1}$$

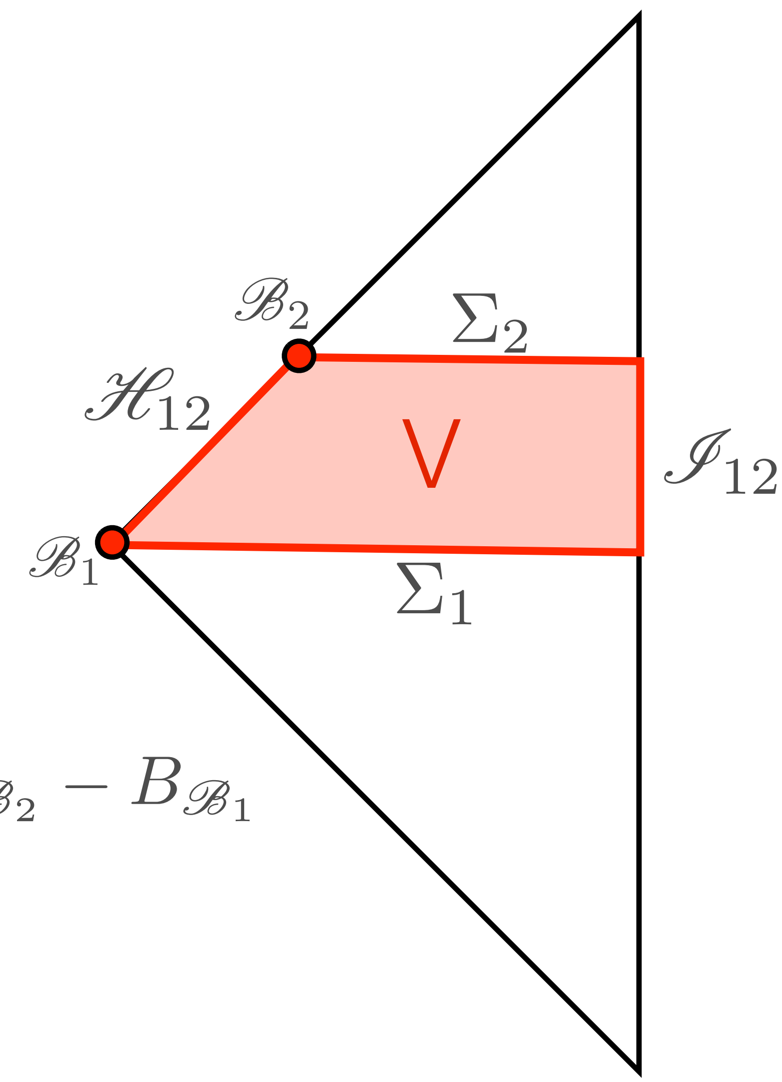
(imposed reflecting AdS boundary, and certain gauge conditions)

Positivity of fluxes

- Integrate over a volume V. On solutions, Stokes' theorem gives

$$0 = \int_V \nabla_a w^a = \int_{\partial V} n_a w^a$$

- Now take $\gamma_2 = \pounds_K \gamma_1$, so $w^a = w^a(\gamma, \pounds_K \gamma)$
and consider contributions from each boundary



$$\boxed{\int_{\mathscr{I}_{12}} n_a w^a = 0}$$

Flux through infinity vanishes in AdS

$$\int_{\mathscr{H}_{12}} n_a w^a = \boxed{\frac{1}{4\pi} \int_{\mathscr{H}_{12}} (K^c \nabla_c u) \delta\sigma_{ab} \delta\sigma^{ab}} + B_{\mathscr{B}_2} - B_{\mathscr{B}_1}$$

nonnegative

(imposed reflecting AdS boundary, and certain gauge conditions)

Canonical energy

- So define the canonical energy

$$\mathcal{E}_K(\gamma, \Sigma) = \int_\Sigma n_a w^a(g; \gamma, \pounds_K \gamma) - B_{\mathscr{B}}(g; \gamma)$$

- Above implies $\mathcal{E}_K(\gamma, \Sigma_2) \leq \mathcal{E}_K(\gamma, \Sigma_1)$ (decreases in time)

- Under restriction to certain gauge conditions at $\mathscr{H}^+$ and $\mathscr{I}$, together with $\delta A = 0$ and $\delta H_X = 0$ for all asymptotic symmetries X^a, it can be shown that $\mathcal{E}_K(\gamma, \Sigma)$ is gauge-invariant and degenerate precisely on perturbations to other stationary black holes.

Stability criterion

Canonical energy non-negative for all such perturbations
STABLE

Canonical energy negative for some perturbation
UNSTABLE

Construction of initial data

- Energy (with respect to K^a) of a particle with 4-momentum p^a is

$$\mathcal{E}_{K,\text{particle}} = -K^a p_a$$

  If there is an ergoregion where $K^a K_a > 0$ is spacelike, then a timelike or null p^a may be chosen to make $\mathcal{E}_{K,\text{particle}} < 0$ in the ergoregion.

- Similarly, for a wave, we ought to be able to find a gravitational perturbation such that the canonical energy $\mathcal{E}_K(\gamma) < 0$

  - **Step 1:** WKB method to obtain approximate compact support solution to the constraint equations of the form $\gamma_{ab} = A_{ab}\exp(i\omega\chi)$ with $\omega \gg 1$ and $\mathcal{E}_K(\gamma) \sim \omega^2 K^a p_a < 0$

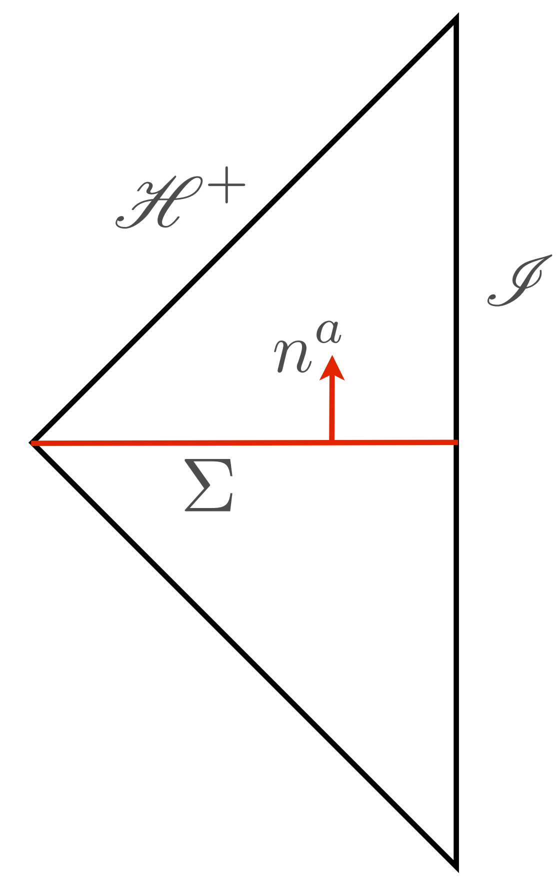
  - **Step 2:** Obtain exact solution with Corvino-Schoen method, such that canonical energy remains negative.

Construction of initial data

- Trade spacetime quantities for initial data quantities defined on Σ

$$g_{ab} \longrightarrow \begin{aligned} q_{ab} &= g_{ab} + n_a n_b \\ p^{ab} &= \sqrt{q}(k^{ab} - q^{ab} k^c{}_c) \end{aligned}$$

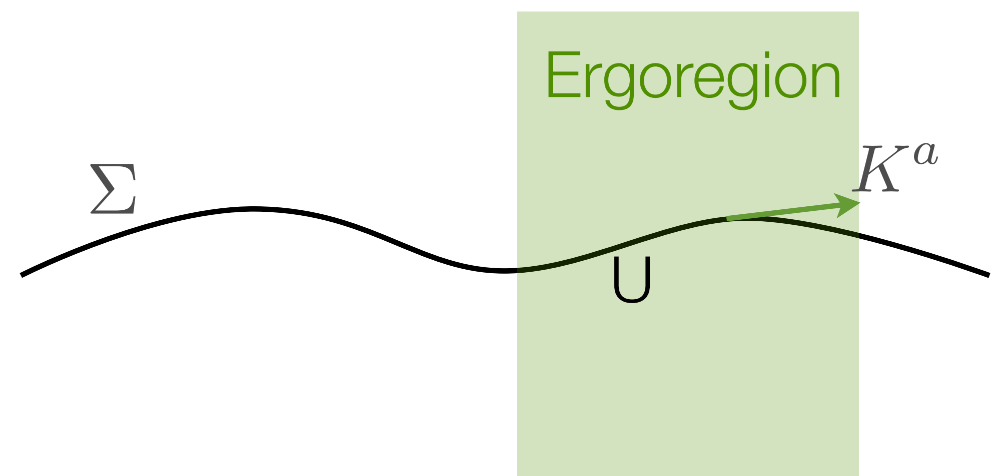
$$\gamma_{ab} \longrightarrow \begin{aligned} \delta q_{ab} &= q_a{}^c q_b{}^d \gamma_{cd} \\ \delta p^{ab} &= \sqrt{q}(q^{ac} q^{bd} - q^{ab} q^{cd}) \frac{1}{2} \pounds_n \gamma_{cd} \end{aligned}$$



Construction of initial data

- Assume there is a region where K^a is spacelike. Construct approximate initial data of compact support in this region.

- Trick: In this region, choose Σ such that it is tangent to K^a (possible since spacelike). This leads to the expression



$$\mathcal{E}_K(\delta q_{ab}, \delta p^{ab}) = -\frac{1}{16\pi}\int_\Sigma K^a \big(-2\delta p^{bc} D_a \delta q_{bc} + 4\delta p^{cb} D_b \delta q_{ac} + 2\delta q_{ac} D_b \delta p^{cb}$$
$$-2p^{cb}\delta q_{ad} D_b \delta q_c{}^d + p^{cb}\delta q_{ad} D^d \delta q_{cb}\big)$$

- Constraints

$$\boldsymbol{C}(\delta q_{ab}, \delta p^{ab}) \equiv \begin{pmatrix} q^{\frac{1}{2}}\left(D^a D_a \delta q_c{}^c - D^a D^b \delta q_{ab} + Ric(q)^{ab}\delta q_{ab}\right) + \\ q^{-\frac{1}{2}}\Big(-\delta q_c{}^c p^{ab} p_{ab} + 2p_{ab}\delta p^{ab} + 2p^{ac} p^b{}_a \delta q_{bc} + \\ \frac{1}{d-2} p^c{}_c p^d{}_d \delta q^a{}_a - \frac{2}{d-2} p^a{}_a \delta p^b{}_b - \frac{2}{d-2}\delta q_{ab} p^{ab} p_c{}^c\Big) \\ \\ -2q^{\frac{1}{2}} D^b(q^{-\frac{1}{2}}\delta p_{ab}) + D_a \delta q_{cb} p^{cb} - 2D_c \delta q_{ab} p^{bc} \end{pmatrix} = 0$$

Construction of initial data

- WKB expansion of initial data $$\delta q_{ab} = \left(\sum_{n \geq 0} Q_{ab}^{(n)} (i\omega)^{-n} \right) \exp(i\omega\chi),$$
$$\delta p_{ab} = \left(\sum_{n \geq 0} P_{ab}^{(n)} (i\omega)^{-n+1} \right) \exp(i\omega\chi)$$

WKB parameter

phase function

- Constraints become

$$\begin{pmatrix} -D^a\chi(D_a\chi)Q_c^{(n)c} + D^a\chi(D^b\chi)Q_{ab}^{(n)} \\ P_{ab}^{(n)} D^b\chi \end{pmatrix} = \boldsymbol{C}^{(n)}$$

Depends on lower order (m<n)
WKB approximations

  - 0th order, choose

$$P_{ab}^{(0)} = -Q_{ab}^{(0)}, \qquad Q_a^{(0)a} = 0, \qquad Q_{ab}^{(0)} D^b\chi = 0$$

  - Higher orders algebraic

Construction of initial data

- To leading order in WKB, the canonical energy is

$$\mathcal{E}(\delta q, \delta p) = -\frac{\omega^2}{16\pi} \int_U K^b D_b \chi \, Q_c^{(0)a} Q_a^{(0)c} + O(\omega)$$

- So choosing $K^a D_a \chi > 0$ gives $\mathcal{E} < 0$ as $\omega \to \infty$

- Of course, any given WKB order is only an approximate solution. Using the Corvino-Shoen method (see paper), we can correct our WKB initial data such that

    - Linearized constraints hold exactly

    - Data remain smooth and compactly supported in slightly larger region

    - The correction to the canonical energy is sufficiently small as $\omega \to \infty$

Conclusions and open questions

- *Any black hole in AdS with a horizon Killing field that becomes spacelike is linearly unstable to superradiant gravitational perturbations.* Results follow from a Lagrangian formulation of the theory, so should carry over to other fields.

- As perturbation grows, nonlinear effects become important:

    - Backreaction of the perturbation on the black hole changes the background.

    - Changing background alters the dynamics of the perturbation. Unstable modes may become stable and fall back into the black hole. [See Bosch, Green and Lehner (2016) for nonlinear results in the charged analog.]

- End point of instability remains unknown. No plausible final state, and numerical simulations are challenging.