A Quantization of Plane Waves: An update

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Plane (Sandwich) Gravitational Waves

Goal: To explore the relation between Lorentz symmetry and discrete spatial geometry

Plane (Sandwich) Gravitational Waves

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- A Lorentzian spacetime with a covariantly constant null vector field $\nabla_a k_b = 0$

- In one choice of chart the metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + L^2 e^{2\beta}\mathrm{d}x^2 + L^2 e^{-2\beta}\mathrm{d}y^2 + \mathrm{d}z^2$$

- Plane wave in one null direction $u = t - z$

- One Einstein equ'n

$$\partial_u^2 L + (\partial_u \beta)^2 L = 0.$$

- Has an exact solution

- Free function β and background factor L recording the warping of space

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$c, \hbar, G, \gamma$ set to unity

In Ashtekar variables $\quad (A_a^i, E^{bj}) \longrightarrow (K_\sigma, E^\rho; \mathcal{A}, \mathcal{E}; \eta, P)$

Constraints of effective 1+1D system: Bojowald, Banerjee, Date, ...

$$G = \mathcal{E}' + P$$

Gauss

$$D = K_x' E^x + K_y' E^y - \mathcal{E}' \mathcal{A} + \eta' P$$

Residual diffeos

$$H \approx -\frac{1}{\sqrt{\mathcal{E} E^x E^y}} \Big\{ E^x K_x E^y K_y + (E^x K_x + E^y K_y)\mathcal{E}\,(\mathcal{A} + \eta') - \frac{1}{4}\mathcal{E}'^2 - \mathcal{E}\mathcal{E}'' - \frac{1}{4}\mathcal{E}^2 \left[\left(\ln \frac{E^y}{E^x}\right)'\right]^2 + \frac{1}{2}\mathcal{E}\mathcal{E}'(\ln E^x E^y)' \Big\}$$

Hamiltonian

$$U = E^x K_x + E^y K_y + \mathcal{E}'$$

Uni-directional

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Algebra: Usual (reduced) GR algebra

Hinterleitner, SM 1106.1448

$$\{G[f],G[g]\}=\{G[f],H[g]\}=0,\ \ \{G[f],D[g]\}=-G[f'g],$$

$$\{D[f],D[g]\}=D[fg'-f'g],\ \ \{D[f],H[g]\}=H[fg'],$$

$$\{H[f],H[g]\}=D\left[(fg'-f'g)\frac{\mathcal{E}}{E^xE^y}\right]$$

With additional uni-directional constraint

$$\{U[f],G[g]\}=0,\ \{U[f],D[g]\}=-U[f'g]\approx 0,$$

$$\{U[f],H[g]\}=-U\left[\sqrt{\frac{\mathcal{E}}{E^xE^y}}\,f'g\right]-H[fg]\approx 0$$

A first class system, with structure functions

Perhaps not too difficult to quantize... But wait, there's more!

Plane (Sandwich) Gravitational Waves

Simplicity due to the dimensional reduction means we have various flavors of the algebra -

By taking appropriate combinations of constraints we have in addition

- As a **Lie Algebra** - no structure functions!

$$\{D[f], \bar{H}[g]\} = \bar{H}[f'g - fg'], \quad \{U[f], \bar{H}[g]\} = U[f'g], \quad \text{with} \quad \bar{H} = \sqrt{\frac{E^x E^y}{\mathcal{E}}} H$$

$$\{\bar{H}[f], \bar{H}[g]\} = D[f'g - fg']$$

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- As an **abelian algebra** (scalar constraint)

$$\{C[f], C[g]\} = 0 \qquad \text{with} \qquad C = \frac{\sqrt{\mathcal{E}}\mathcal{E}'}{E^x E^y}(H' + D)$$

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- As an algebra where the **Hamiltonian constraint forms an ideal**

$$\{D[f], \tilde{H}[g]\} = \tilde{H}[f'g - fg'], \quad \{\tilde{H}[f], \tilde{H}[g]\} = 8\tilde{H}[f'g - fg']$$

Which algebra?

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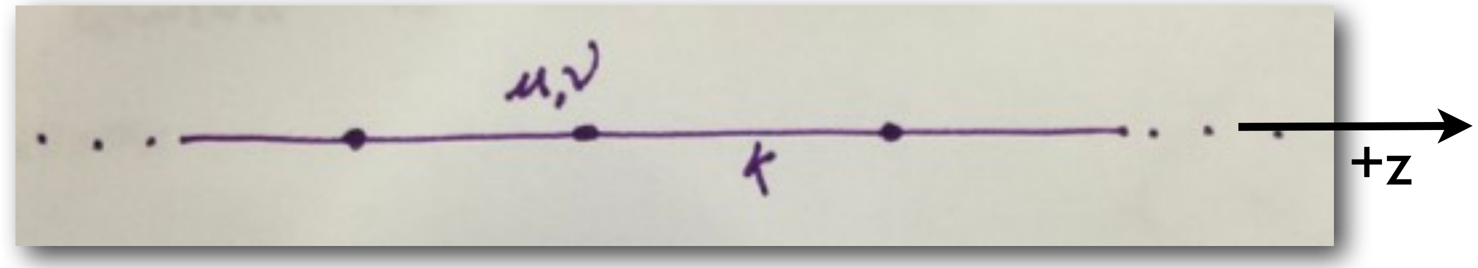
Banerjee, Date 0712.0687

Kinematics: States based on 1D graph



spin network

For the $U(1)$ connection

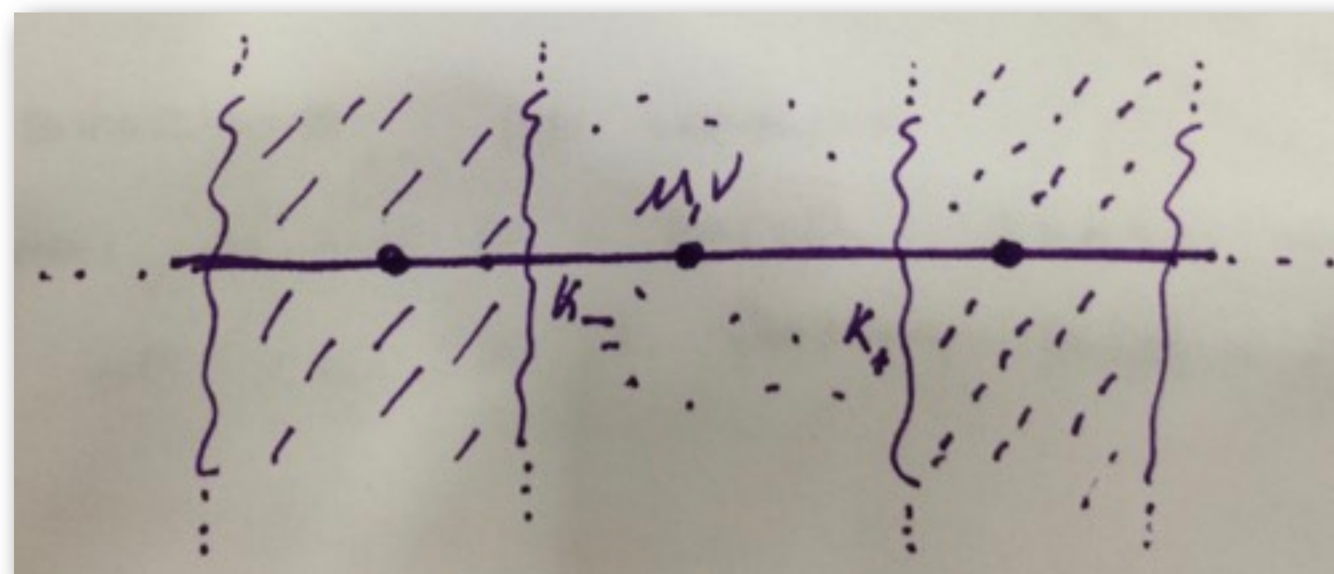
$$h_e(\mathcal{A}) = e^{\,i\frac{k}{2}\int_e \mathcal{A}}$$

For scalars e.g. K_x $\quad h_v(K_x) = e^{\,i\frac{\mu}{2}K_x(v)}$

Geometric quantities are simple

$$\widehat{\mathcal{E}}\,|\mu,\nu,k\rangle = \frac{k}{2}|\mu,\nu,k\rangle$$

$$\int_I \widehat{E^x}\,|\mu,\nu,k\rangle = \frac{\mu}{2}|\mu,\nu,k\rangle$$



Volume:

$$\hat{V}\,|\mu,\nu,k\rangle = \frac{1}{4}\sqrt{|\mu|\,|\nu|\,|k_+ + k_-|}\,|\mu,\nu,k\rangle$$

$|\mu,\nu,k\rangle$ an "atom of geometry"

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A choice of algebra that allows us to check... Varadarajan, Tomlin, Laddha 1105.0636,....

How do we know that we have the correct quantum theory as checked via the classical limit? If we quantize diffeo and hamiltonian constraints on same footing then we can check the algebra of constraints.

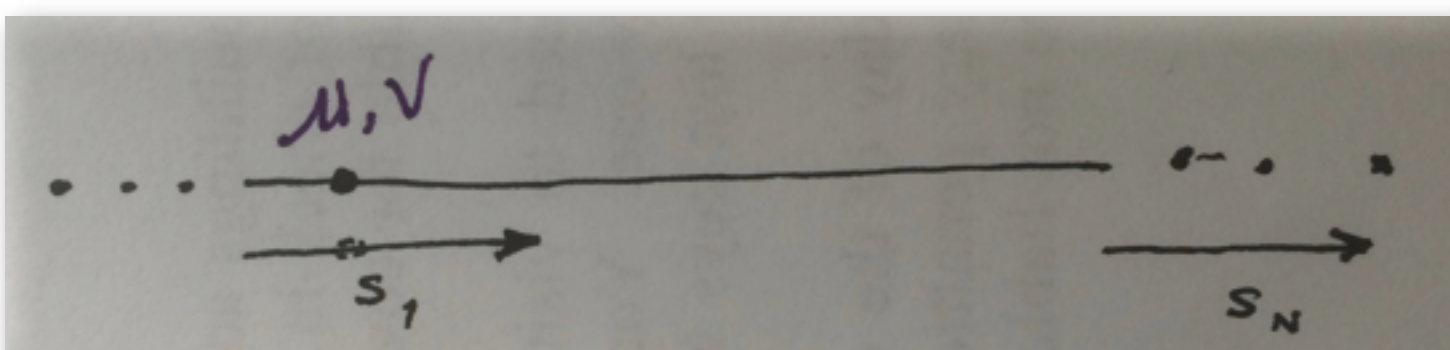
One of these versions of the algebra allows checking of the algebra, but without structure functions.

With triangulation adapted to the shift f can a diffeomorphism Φ be expressed as

$$\left(1 + i\delta \widehat{D_T}[f]\right) h_I = h_{\Phi(f,\delta)\circ e} \quad ?$$

Yes. It is a shift operator

$$\left(1 + \frac{i\delta}{\ell_P^2 \gamma} \widehat{D_T}[f]\right) h_I = h_v^{-1}[\vec{A}]\, h_{s_1}^{-1}[\mathcal{A}] h_I[\vec{A}, \mathcal{A}] h_{v'}[\vec{A}]\, h_{s_N}[\mathcal{A}] = h_{\Phi(f,\delta)\circ I}$$



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δ parameterizes both the triangulation and diffeomorphism

How about the Hamiltonian constraint?

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Hamiltonian constraint is apparently not so simple: Action on an atom of geometry

$$\hat{H}\,|\vec{v}\rangle = -f(\vec{v})\,|\vec{v}\rangle + ... + \frac{l_P}{8\gamma^{\frac{3}{2}}\mu_0\nu_0}\sqrt{\mu\nu}\left(\sqrt{k_- + k_+ + 1} - \sqrt{k_- + k_+ - 1}\right)\Big(|\mu_v + 2\mu_0, \nu_v - 2\nu_0\rangle$$

$$+ |\mu_v - 2\mu_0, \nu_v + 2\nu_0,\rangle - |\mu_v + 2\mu_0, \nu_v + 2\nu_0\rangle - |\mu_v - 2\mu_0, \nu_v - 2\nu_0\rangle\Big)$$

$$+ \frac{l_P}{32\gamma^{\frac{3}{2}}\mu_0\nu_0}\sum_v \sqrt{|\mu_v|\,|k_+ + k_-|}\left(\sqrt{\nu_v + \nu_0} - \sqrt{\nu_v - \nu_0}\right)\big[\,|\mu_v + \mu_0, k_+ + 1, \mu_{v+1} - \mu_0\rangle$$

$$- |\mu_v + \mu_0, k_+ + 1, \mu_{v+1} + \mu_0\rangle + |\mu_v + \mu_0, k_+ - 1, \mu_{v+1} + \mu_0\rangle - |\mu_v + \mu_0, k_+ - 1, \mu_{v+1} - \mu_0\rangle$$

$$+ |\mu_v - \mu_0, k_+ + 1, \mu_{v+1} - \mu_0\rangle - |\mu_v - \mu_0, k_+ + 1, \mu_{v+1} + \mu_0\rangle + |\mu_v - \mu_0, k_+ - 1, \mu_{v+1} + \mu_0\rangle$$

$$- |\mu_v - \mu_0, k_+ - 1, \mu_{v+1} - \mu_0\rangle - |\mu_v + \mu_0, k_- + 1, \mu_{v-1} - \mu_0\rangle + |\mu_v + \mu_0, k_- + 1, \mu_{v-1} + \mu_0\rangle$$

$$- |\mu_v + \mu_0, k_- - 1, \mu_{v-1} + \mu_0\rangle + |\mu_v + \mu_0, k_- - 1, \mu_{v-1} - \mu_0\rangle - |\mu_v - \mu_0, k_- + 1, \mu_{v-1} - \nu_0\rangle$$

$$+ |\mu_v - \mu_0, k_- + 1, \mu_{v-1} + \mu_0\rangle - |\mu_v - \mu_0, k_- - 1, \mu_{v-1} + \mu_0\rangle$$

$$+ |\mu_v - \mu_0, k_- - 1, \mu_{v-1} - \mu_0\rangle\,\big] + \frac{l_P}{32\gamma^{\frac{3}{2}}\mu_0\nu_0}\sum_v \sqrt{|\mu_v|\,|k_+ + k_-|}\left(\sqrt{\nu_v + \nu_0} - \sqrt{\nu_v - \nu_0}\right)$$

$$\times\big[\,|\mu_v + \mu_0, k_+ + 1, \mu_{v+1} - \mu_0\rangle - |\mu_v + \mu_0, k_+ + 1, \mu_{v+1} + \mu_0\rangle$$

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Plane Gravitational Waves LQG

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$$
\begin{aligned}
\hat{H}\,|\vec{v}\rangle &= -f(\vec{v})\,|\vec{v}\rangle + ... + \frac{l_P}{8\gamma^{\frac{3}{2}}\mu_0\nu_0}\sqrt{\mu\nu}\left(\sqrt{k_- + k_+ + 1} - \sqrt{k_- + k_+ - 1}\right)\Big(|\mu_v + 2\mu_0, \nu_v - 2\nu_0\rangle \\
&+ |\mu_v - 2\mu_0, \nu_v + 2\nu_0, \rangle - |\mu_v + 2\mu_0, \nu_v + 2\nu_0\rangle - |\mu_v - 2\mu_0, \nu_v - 2\nu_0\rangle\Big) \\
&+ \frac{l_P}{32\gamma^{\frac{3}{2}}\mu_0\nu_0}\sum_v \sqrt{|\mu_v|\,|k_+ + k_-|}\left(\sqrt{\nu_v + \nu_0} - \sqrt{\nu_v - \nu_0}\right)\big[\,|\mu_v + \mu_0, k_+ + 1, \mu_{v+1} - \mu_0\rangle \\
&- |\mu_v + \mu_0, k_+ + 1, \mu_{v+1} + \mu_0\rangle + |\mu_v + \mu_0, k_+ - 1, \mu_{v+1} + \mu_0\rangle - |\mu_v + \mu_0, k_+ - 1, \mu_{v+1} - \mu_0\rangle \\
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&\times \big[\,|\mu_v + \mu_0, k_+ + 1, \mu_{v+1} - \mu_0\rangle - |\mu_v + \mu_0, k_+ + 1, \mu_{v+1} + \mu_0\rangle \\
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\end{aligned}
$$

The Physics is Opaque

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But there are better ways to motivate quantization choices:
What we actually have is in a constraint terms of dilitation P_Δ and shear P_β in x,y plane

$$H = \frac{P_\beta^2 + P_\Delta^2}{4\mathcal{E}} + P_\Delta \mathcal{A} + \frac{1}{4}\frac{\mathcal{E}'^2}{\mathcal{E}} + \mathcal{E}'' + \mathcal{E}\beta'^2 - \mathcal{E}'\Delta'$$

$$D = -P_\Delta' + \beta' P_\beta + \Delta' P_\Delta - \mathcal{E}'\mathcal{A}$$

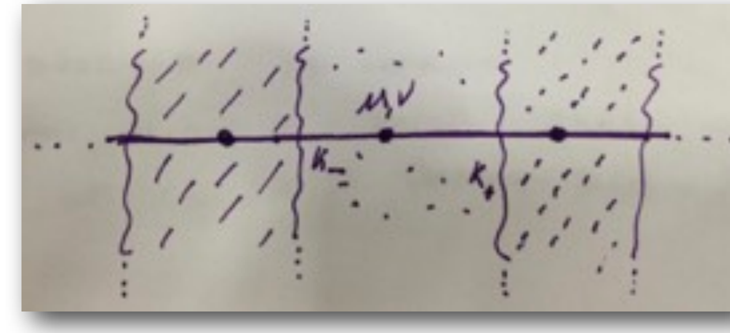
$$U = P_\Delta + \mathcal{E}'$$

with

$$P_\beta = E^y K_y - E^x K_x$$
$$-P_\Delta = E^x K_x + E^y K_y$$

- quantization is already partially based on these operator products
- easier to use classical solutions: β is as before, both $\mathcal{E}$ and Δ are related to warping factor
- work is ongoing...

Summary: "A Quantization of Plane Waves: An Update"



0. Kinematics and geometric operators complete, e.g.

$$\hat{V}\,|\mu,\nu,k\rangle = \frac{1}{4}\sqrt{|\mu|\,|\nu|\,|k_+ + k_-|}\,|\mu,\nu,k\rangle$$

1. Use form of constraint algebra (H, D, U) without structure functions

$$\{D[f],\bar{H}[g]\} = \bar{H}[f'g - fg'], \quad \{U[f],\bar{H}[g]\} = U[f'g], \quad \{\bar{H}[f],\bar{H}[g]\} = D[f'g - fg']$$

2. Quantize diffeo and Hamiltonian constraints on same footing

$$\left(1 + \frac{i\delta}{\ell_P^2\gamma}\widehat{D_T}[f]\right) h_I = h_v^{-1}[\vec{A}]\, h_{s_1}^{-1}[\mathcal{A}] h_I[\vec{A},\mathcal{A}] h_{v'}[\vec{A}]\, h_{s_N}[\mathcal{A}] = h_{\Phi(f,\delta)\circ I}$$

3. Quantize Hamilton constraint using factors that have physical interpretation, e.g. shear and diliatation

4. ... investigate role of discreteness and Lorentz symmetry