Interpretation of Generic Off-Diagonal Exact Solutions in Einstein Gravity and Modifications

Sergiu I. **Vacaru**
joint work: Klee **Irwin**, Laurenţiu **Bubuianu**, Olivia **Vacaru**

QGR, Topanga, USA & Project IDEI (UAIC), Iaşi, Romania

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Exact solutions and their interpretation
Chair: Jiri Podolsky (Charles University, Czech Republic)

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Motivations and Goals:

1. Standard Methods for constructing important physical solutions in GR: nonlinear systems PDEs $\rightarrow$ nonlinear systems ODEs
   For BH, solutions parameterized by 2 integration constants, $g_{\alpha}(r, c_1, c_2)$ diagonal ansatz, prescribed symmetries, asymptotic conditions ...
2. Construct (physical importance?) of generic off-diagonal solutions depending on all spacetime coordinates, six independent

$$g_{\alpha\beta}(u^{\gamma}), \alpha = 1, 2, 3, 4;$$

   Physical and geometric interpretation of nonlinear parametric solutions of PDEs determined by generating and integration functions?
3. Modeling MGTs effects in GR and geometric evolution scenarios by generic off-diagonal interactions and nonholonomic constraints. Nontrivial vacuum structure, cosmological acceleration and dark energy/matter effects.

Nonholonomic 2+2+2+... splitting

Aims: Find $e_\alpha = e_\alpha^{\alpha'} \partial_{\alpha'}$ when Einstein eqs for $(\mathbf{g}, \mathbf{D})$ with a generalized / auxiliary connection $\mathbf{D}$ decouple and can be integrated in very general forms.

Non–integrable (nonholonomic) 2+2 spacetime splitting in GR $(V, \mathbf{g})$,

4–d pseudo–Riemannian V, $\mathbf{g} = g_{\alpha\beta}$ with conventional $2+2$ splitting:
indices $\alpha, \beta, ... = (i, a), (j, b), ...$ for $i, j, k, ... = 1, 2$; $a, b, c, .. = 3, 4$;
coordinates $u^\alpha = (x^i, y^a) = (x^1, x^2, y^3, y^4)$, or $u = (x, y)$,
partial derivatives $\partial_\alpha := \partial / \partial u^\alpha$; $\partial_\alpha = (\partial_i, \partial_a)$

N–adapted frames/ bases: $\mathbf{N} : TV = hTV \oplus vTV$; $\mathbf{N} = N_i^a(x, y) \partial_a \otimes dx^i$

nonholonomic frames: $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = \mathbf{w}^\gamma{}_{\alpha\beta}(u)\, \mathbf{e}_\gamma$,

$$
\begin{aligned}
\mathbf{e}_\alpha &:= (\mathbf{e}_i = \partial_i - N_i^a \partial_a, e_b = \partial_b) \\
\mathbf{e}^\beta &:= (e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a dx^i)
\end{aligned}
$$

anholonomy coefficients $\mathbf{w}^\gamma{}_{\alpha\beta}(u)$ are functionals of $N_i^a(x, y)$ and part deriv

two preferred linear connections

$$\mathbf{g} \rightarrow \left\{ \begin{array}{lll} \nabla : & \nabla \mathbf{g} = 0;\ {}^{\nabla}\mathcal{T}^{\alpha} = 0, & \text{the Levi–Civita connection ;} \\ \widehat{\mathbf{D}} : & \widehat{\mathbf{D}}\mathbf{g} = 0;\ h\widehat{\mathcal{T}}^{\alpha} = 0, v\widehat{\mathcal{T}}^{\alpha} = 0, & \text{the canonical d–connection} \end{array} \right.$$

$$\widehat{\mathbf{D}}[\mathbf{g}] = \nabla[\mathbf{g}] + \widehat{\mathbf{Z}}[\mathbf{g}]$$

$\exists$ different classes of linear connections completely determined by a metric $\mathbf{g}$
idea taken from Lagrange–Finsler geometry y^a are fiber coordinates on TM.

"auxiliary" connection $\widehat{\mathbf{D}} = \widehat{\Gamma}^{\gamma}_{\ \alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})$ in N–adapted frames,

$\widehat{L}^i_{jk} = \frac{1}{2} g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}),$
$\widehat{L}^a_{bk} = e_b(N^a_k) + \frac{1}{2} g^{ac}(e_k g_{bc} - g_{dc}\ e_b N^d_k - g_{db}\ e_c N^d_k),$
$\widehat{C}^i_{jc} = \frac{1}{2} g^{ik} e_c g_{jk},\ \widehat{C}^a_{bc} = \frac{1}{2} g^{ad}(e_c g_{bd} + e_c g_{cd} - e_d g_{bc})$

1) $\widehat{\mathbf{D}}\mathbf{g} = 0$, 2) $\widehat{T}^i_{\ jk} = 0,\ \widehat{T}^a_{\ bc} = 0.$

Anholonomic Torsion $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}:\ \widehat{T}^i_{\ ja} = \widehat{C}^i_{jb},\ \widehat{T}^a_{\ ji} = -\Omega^a_{\ ji},\ \widehat{T}^c_{\ aj} = \widehat{L}^c_{aj} - e_a(N^c_j).$

With respect to N–adapted frames and for $\widehat{\mathbf{D}}$

N–adapted Einstein eqs: $$\widehat{\mathbf{R}}_{\ \beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\ {}^{s}R = \Upsilon_{\beta\delta},$$

LC–conditions for GR: $$\widehat{L}^{c}_{aj} = e_a(N^c_j),\ \widehat{C}^{i}_{jb} = 0,\ \Omega^{a}_{\ ji} = 0,$$

$\widehat{\mathbf{R}}_{\ \beta\delta}$ for $\widehat{\Gamma}^{\gamma}_{\ \alpha\beta}$, ${}^{s}R = \mathbf{g}^{\beta\delta}\widehat{\mathbf{R}}_{\ \beta\delta}$
and $\Upsilon_{\beta\delta} \to \varkappa T_{\beta\delta}$; iff $\widehat{\mathbf{D}} \to \nabla$ result in (effective) GR

Goals: 1. Prove that such equations can be integrated in very general off–diagonal forms depending on all spacetime coordinates via generating and integration functions and various type parameters.

2. Extract generic off–diagonal solutions in GR
3. Extract off–diagonal solutions in higher dimension gravity (5-10-d)
4. Mimic by off–diagonal interactions in GR various STRING, TMT and MGTs

2+2 conventional splitting (x^i, y^a), for $i = 1,2$ and $a = 3,4$, with $y^4 = t$,"

$$\mathbf{g}_{\alpha\beta} = \left| \begin{array}{cccc} g_1 + \omega^2({n_1}^2 h_3 + {w_1}^2 h_4) & \omega^2(n_1 n_2 h_3 + w_1 w_2 h_4) & \omega^2\ n_1 h_3 & \omega^2\ w_1 h_4 \\ \omega^2(n_1 n_2 h_3 + w_1 w_2 h_4) & g_2 + \omega^2({w_2}^2 h_3 + {n_2}^2 h_4) & \omega^2\ n_2 h_3 & \omega^2\ w_2 h_4 \\ \omega^2\ n_1 h_3 & \omega^2\ n_2 h_3 & \omega^2 h_3 & 0 \\ \omega^2\ w_1 h_4 & \omega^2\ w_2 h_4 & 0 & \omega^2 h_4 \end{array} \right|$$

N–adapted frames of reference

$$\begin{aligned} \mathbf{g} &= g_i dx^i \otimes dx^i + \omega^2 h_a \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + n_i dx^i,\ \mathbf{e}^4 = dt + w_i dx^i, \end{aligned}$$

$g_i = g_i(x^k), g_a = \omega^2(x^i, y^c)\ h_a(x^k, t)$, not summation on "a";
$N_i^3 = n_i(x^k, t), N_i^4 = w_i(x^k, t)$

are functions of necessary smooth class generating solutions of grav. field eqs.

canonical d–connection: $\widehat{\mathbf{D}}\mathbf{g} = 0$; $h\widehat{\mathbf{T}},\ v\widehat{\mathbf{T}} = 0,\ hv\widehat{\mathbf{T}} \neq 0$

$\widehat{\mathbf{D}}$ completely determined by $\mathbf{g}$ but not the LC–connection un–known functions $\psi(x^i), h_a(x^k,t), w_i(x^k,t)$ and $n_i(x^k,t)$,

where $\psi^\bullet = \partial_1\psi, \psi' = \partial_2\psi, \varpi^* = \partial_4\varpi$; generating function $\Psi := e^\varpi$

$$\begin{aligned} \psi^{\bullet\bullet} + \psi'' &= 2\ \widetilde{\Upsilon} \\ \varpi^* h_3^* &= 2h_3 h_4 \Upsilon \\ n_i^{**} + \gamma n_i^* &= 0 \\ \beta w_i - \alpha_i &= 0 \end{aligned}$$

$$\varpi = \ln\left|\frac{\partial_4 h_3}{\sqrt{|h_3 h_4|}}\right|,\ \gamma := \partial_4(\ln\frac{|h_3|^{3/2}}{|h_4|}), \qquad \alpha_i = \frac{\partial_4 h_3}{2h_3}\partial_i\varpi,\ \beta = \frac{\partial_4 h_3}{2h_3}\partial_4\varpi$$

for v-conformal factor ω, $\qquad \varpi^* w_i - \partial_i\varpi = 0$ and $\varpi^*\partial_i\omega - \omega^*\partial_i\varpi = 0$

additional constraints to satisfy the torsionless conditions

$$\begin{aligned} w_i^* &= (\partial_i - w_i\partial_4)\ln\sqrt{|h_4|}, (\partial_i - w_i\partial_4)\ln\sqrt{|h_3|} = 0, \partial_i w_j = \partial_j w_i, \\ n_i^* &= 0, \partial_i n_j = \partial_j n_i. \end{aligned}$$

Off–diagonal "one–Killing" solutions

Theorem (Integral Varieties): generating functs $\psi(x^k,\theta); \varpi(x^k,t,\theta)$, $\Psi := e^{\varpi}$;

integration functions $\ {}^0h_a(x^k,\theta),\ {}_1n_k(x^k,\theta),\ {}_2n_k(x^k,\theta);$ source $\Upsilon(x^k,t,\theta)$

$$g_i = \varepsilon_i e^{\psi},\ h_3[\widetilde{\Psi},\Lambda_0] = \frac{\widetilde{\Psi}^2}{4\Lambda_0} \text{ and } h_4[\widetilde{\Psi},\Lambda_0,\Xi] = \frac{(\widetilde{\Psi}^*)^2}{\Xi},$$

$$n_k = {}_1n_k + {}_2n_k \int dt \frac{h_4}{(\sqrt{|h_3|})^3} = {}_1n_k + {}_2\widetilde{n}_k \int dt \frac{(\widetilde{\Psi}^*)^2}{\widetilde{\Psi}^3\ \Xi},$$

$$w_i = \frac{\partial_i \varpi}{\varpi^*} = \frac{\partial_i \Psi}{\Psi^*} = \frac{\partial_i \Psi^2}{(\Psi^2)^*} = \int dt \frac{\partial_i[\Upsilon(\widetilde{\Psi}^2)^*]}{[\Upsilon(\widetilde{\Psi}^2)]^*} = \frac{\partial_i \Xi}{\Xi^*},$$

an effective cosm. constant $\Lambda_0 \neq 0$ re–defined the generating funct. $\Psi \longleftrightarrow \widetilde{\Psi}$,

$\Psi^2 = \Lambda_0^{-1} \int dt \Upsilon \partial_t(\widetilde{\Psi}^2)$ and $\widetilde{\Psi}^2 = \Lambda_0 \int dt \Upsilon^{-1} \partial_t(\Psi^2)$, $\Xi[\Upsilon,\widetilde{\Psi}] = \int dt \Upsilon \partial_t(\widetilde{\Psi}^2)$

Solutions determined by generating functions and effective sources

$\left[\psi(x^k), \widehat{\Psi}(x^k,t), \omega(x^k,y^3,t)\right]$, $\widehat{\Xi}(x^k,t)$, parameter Λ_0, and

integration functions ${}_1n_i = \partial_i n(x^k)$.

NONTRIVIAL ANHOLONOMY COEFFICIENTS - not coordinate transforms

Properties of Generating and Integration Functions and Sources

Important property: $\Lambda \Psi^2 = \widetilde{\Psi}^2 |\Upsilon| + \int dt \widetilde{\Psi}^2 |\Upsilon|^* \ \Psi \longleftrightarrow \widetilde{\Psi}$, when $\Lambda(\Psi^2)^* = |\Upsilon|(\widetilde{\Psi}^2)^*, \Psi^* \neq 0$

for $\widetilde{\Psi} := \exp \widetilde{\varpi}$ and any prescribed values of effective (for different types of contributions ef, m, f, μ) cosmological constants in $\Lambda = {}^{ef}\Lambda + {}^{m}\Lambda + {}^{f}\Lambda + {}^{\mu}\Lambda$ associated to

$$\Upsilon(x^k, t) = {}^{ef}\Upsilon(x^k, t) + {}^{m}\Upsilon(x^k, t) + {}^{f}\Upsilon(x^k, t) + {}^{\mu}\Upsilon(x^k, t)$$

and effective source $\Xi := \int dt \Upsilon(\widetilde{\Psi}^2)^* = {}^{ef}\Xi + {}^{m}\Xi + {}^{f}\Xi + {}^{\mu}\Xi$,
when ${}^{ef}\Xi := \int dt\ {}^{ef}\Upsilon(\widetilde{\Psi}^2)^*,\ {}^{m}\Xi := \int dt\ {}^{m}\Upsilon(\widetilde{\Psi}^2)^*,\ {}^{f}\Xi := \int dt\ {}^{f}\Upsilon(\widetilde{\Psi}^2)^*$

inhomogeneous cosmological solutions are determined by corresponding classes of

generating functions:	$\psi(x^k), \widetilde{\Psi}(x^k, t), \omega(x^k, y^3, t)$
effective sources:	$\widetilde{\Upsilon}(x^k);\ {}^{ef}\Xi(x^k, t),\ {}^{m}\Xi(x^k, t),\ {}^{f}\Xi(x^k, t),\ {}^{\mu}\Xi(x^k, t)$,
	or ${}^{ef}\Upsilon(x^k, t),\ {}^{m}\Upsilon(x^k, t),\ {}^{f}\Upsilon(x^k, t),\ {}^{\mu}\Upsilon(x^k, t)$
integration cosmological constants:	${}^{ef}\Lambda,\ {}^{m}\Lambda,\ {}^{f}\Lambda,\ {}^{\mu}\Lambda$
integration functions:	${}_1n_i(x^k)$ and ${}_2n_i(x^k)$

generate solutions with any (non)trivial ${}^{ef}\Lambda,\ {}^{m}\Lambda,\ {}^{f}\Lambda,\ {}^{\mu}\Lambda$ or effective sources ${}^{ef}\Upsilon,\ {}^{m}\Upsilon,\ {}^{f}\Upsilon,\ {}^{\mu}\Upsilon$

Modifications of GR: $\nabla[\mathbf{g}] \to \mathbf{D}[\mathbf{g}]$; Lagrange density $R \to f(R,T)$

Vacuum MG: $f_R \mathbf{R}_{\alpha\beta} - \frac{1}{2} f \mathbf{g}_{\alpha\beta} + (\mathbf{g}_{\alpha\beta}\mathbf{D}_\gamma \mathbf{D}^\gamma - \mathbf{D}_\alpha \mathbf{D}_\beta) f_R = 0,$

for $f_R = \partial f/\partial R$. If $\mathbf{D} = \nabla$, vacuum $f(R)$ gravity.

generalized Ricci solitons: $\mathbf{R}_{\alpha\beta} + \mathbf{D}_\alpha \mathbf{D}_\beta K = \lambda \mathbf{g}_{\alpha\beta},$

$K = f_R$ and $\mathbf{D} \to \nabla$ and $\mathbf{g} \to \widehat{\mathbf{g}}$; stationary geometric flows
generalized Einstein spaces: $\mathbf{g} \to \widehat{\mathbf{g}}[\Phi]$; second measure Φ

MG with effective cosmological "constants" & field eqs

$$\mathbf{R}_{\alpha\beta} = \Lambda(x^i, y^a)\mathbf{g}_{\alpha\beta},$$

"Polarized" cosmological constant $\Lambda = \frac{\lambda + \mathbf{D}_\gamma \mathbf{D}^\gamma f_R - f/2}{1 - f_R}.$

Generic off–diagonal solutions for Killing symmetry on $\partial/\partial y^3$, $\Lambda \approx \Lambda(x^i)$.

A. S. Haupt, O. Lechtenfeld and E. T. Musaev, JHEP 1411 (2014) 152

$$\begin{aligned}
\widehat{\mathbf{R}}_{\mu_s\nu_s} + 2({}^s\widehat{\mathbf{D}}\widehat{\mathbf{d}}\widehat{\phi})_{\mu_s\nu_s} - \frac{1}{4}\widehat{\mathbf{H}}_{\alpha_s\beta_s\mu_s}\widehat{\mathbf{H}}_{\nu_s}{}^{\alpha_s\beta_s} + & \\
\frac{\alpha'}{4}\left[\widetilde{\mathbf{R}}_{\mu_s\alpha_s\beta_s\gamma_s}\widetilde{\mathbf{R}}_{\nu_s}{}^{\alpha_s\beta_s\gamma_s} - tr\left(\widehat{\mathbf{F}}_{\mu_s\alpha_s}\widehat{\mathbf{F}}_{\nu_s}{}^{\alpha_s}\right)\right] &= 0, \\
{}^s\widehat{R} + 4\widehat{\square}\widehat{\phi} - 4|\widehat{\mathbf{d}}\widehat{\phi}|^2 - \frac{1}{2}|\widehat{\mathbf{H}}|^2 + \frac{\alpha'}{4} tr\left[|\widetilde{\mathbf{R}}|^2 - |\widehat{\mathbf{F}}|\right] &= 0, \\
e^{2\widehat{\phi}}\widehat{\mathbf{d}}\widehat{*}(e^{-2\widehat{\phi}}\widehat{\mathbf{F}}) + \widehat{\mathbf{A}} \wedge \widehat{*}\widehat{\mathbf{F}} - \widehat{*}\widehat{\mathbf{F}} \wedge \widehat{\mathbf{A}} + \widehat{*}\widehat{\mathbf{H}} \wedge \widehat{\mathbf{F}} &= 0, \\
\widehat{\mathbf{d}}\widehat{*}(e^{-2\widehat{\phi}}\widehat{\mathbf{H}}) &= 0,
\end{aligned}$$

$\widehat{*}$, ${}^s\widehat{\mathbf{D}} = \{\widehat{\mathbf{D}}_{\mu_s}\}$, $\widehat{\square} := \widehat{\mathbf{g}}^{\mu_s\nu_s}\widehat{\mathbf{D}}_{\mu_s}\widehat{\mathbf{D}}_{\nu_s}$, $\widehat{\mathbf{R}}_{\mu_s\nu_s}$, ${}^s\widehat{R}$; $\widetilde{\mathbf{R}}_{\mu_s\alpha_s\beta_s\gamma_s}$ for almost Kähler structure
gauge field $\widehat{\mathbf{A}}$ corresponds to ${}^s_A\widehat{\mathbf{D}} = {}^s\widehat{\mathbf{D}} + \psi_2(y^4)[\mathbf{e}^{a_1} I_{a_1} + \mathbf{e}^{a_2} I_{a_2} + \mathbf{e}^{a_3} I_{a_3}] = \widehat{\mathbf{d}} + \widehat{\mathbf{A}}$
curvature $\widehat{\mathbf{F}} = \mathcal{F}(\psi_2)$ via a map for ${}^s\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}=0} \to {}^s\nabla$

Goals: Integrate in off-diagonal forms for all 10 coordinates (not warped / compactified); MGTs & Geometric Flows.

Nonholonomic deformations 'prime' $\to$ 'target' metrics

Relation to a well known and physically important solution $^\star\mathbf{g}$.

Dependence on y^3, "vertical" conformal $\omega^2(x^j, y^3, t), \partial a/\partial y^3 := a^\circ$,

$\omega^2 = 1$ results in solutions with Killing symmetry,

$$\begin{aligned}
\mathbf{g} &= g_i(x^k)dx^i \otimes dx^i + \omega^2(x^j, y^3, t)h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a, \\
\mathbf{e}^3 &= dy^3 + w_i(x^k, t)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k, t)dx^i, \\
\mathbf{e}_k\omega &= \partial_k\omega + n_k\omega^\circ + w_k\omega^* = 0.
\end{aligned}$$

N–deformations & gravitational polarizations η_α, η_i^a,

N–deforms, $^\star\mathbf{g} = [\,^\star g_i, \,^\star h_a, \,^\star N_k^a] \to \,^\eta\mathbf{g} = [\, g_i, h_a, N_k^a]$,

$$^\eta\mathbf{g} = \eta_i(x^k, t)\,^\star g_i(x^k, t)dx^i \otimes dx^i + \eta_a(x^k, t)\,^\star h_a(x^k, t)\mathbf{e}^a \otimes \mathbf{e}^a,$$

$$\mathbf{e}^3 = dy^3 + \eta_i^3(x^k, t)\,^\star w_i(x^k, t)dx^i, \ \mathbf{e}^4 = dt + \eta_i^4(x^k, t)\,^\star n_i(x^k, t)dx^i.$$

For a solution in GR with well–defined boundary/ asymptotic conditions, we can search $^\star\mathbf{g} \to \,^\eta\mathbf{g}$ to a "parametric/noncommutative/stochastic ..." solution in GR, MG.

$$\begin{aligned} ds^2 &= \mathring{g}_{\alpha'\beta'}(x^{k\prime})du^{\alpha'}du^{\beta'} \text{ ("prime" metric)} \\ &= (1-\frac{M}{r}+\frac{K}{r^2})^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\varphi^2 - (1-\frac{M}{r}+\frac{K}{r^2})dt^2 \\ &= \mathring{g}_{1'}(dx^{1'})^2 + \mathring{g}_{2'}(x^{1'})(dx^{2'})^2 + \mathring{h}_{3'}(x^{1'},x^{2'})(dy^{3'})^2 + \mathring{h}_{4'}(x^{1'})(dy^{4'})^2, \end{aligned}$$

constants M and K

$$\begin{aligned} x^{1'} &= \int dr(1-\frac{M}{r}+\frac{K}{r^2})^{-1/2}, x^{2'}=\theta, y^{3'}=\varphi, y^{4'}=t; \\ \mathring{g}_{1'} &= 1, \mathring{g}_{2'}(x^{1'}) = r^2(x^{1'}), \mathring{h}_{3'} = r^2(x^{1'})\sin(x^{2'}), \mathring{h}_{4'} = -(1-\frac{M}{r(x^{1'})}+\frac{K}{r^2(x^{1'})}) \end{aligned}$$

open region $U \subset V$, where $x^{1'}(r)$ allows to find $r(x^{1'})$ in a unique form.
This metric was studied as a spherical symmetric vacuum solution in R^2 gravity (in our approach, of the Ricci soliton equations.
solution does not exist if $R=0$ (for LC-configurations); not allowed by conformal transforms

Asymptotically de Sitter black ellipsoids in R^2 gravity as Ricci solitons; ε–deformations

For $\underline{\mathring{g}}_4 = \underline{\mathring{h}}_4(\widetilde{x}^{1'}) = (1 - \frac{M}{r} + \lambda r^2)$
and $(\partial_4 t)^2 = 1$ and anisotropically polarized mass
$\widetilde{M}(\varphi) = M[1 + \varepsilon \cos(\omega_0 \varphi + \varphi_0)]$,

$$
\begin{aligned}
{}^s h_4 &= -(1 - \frac{M}{r} + \lambda r^2)[1 - \varepsilon \frac{M}{r} \frac{\cos(\omega_0 \varphi + \varphi_0)}{1 - \frac{M}{r} + \lambda r^2}] \\
&= \underline{\mathring{h}}_4(\widetilde{x}^{1'}) \left[1 - \varepsilon \frac{M}{r} (\underline{\mathring{h}}_4)^{-1} \cos(\omega_0 \varphi + \varphi_0)\right] \simeq - \left[1 - \frac{\widetilde{M}(\varphi)}{r} + \lambda r^2\right]
\end{aligned}
$$

The parametric equation of an ellipse with radial parameter $\mathring{r}_+ = M$ and eccentricity ε,

$$
r_+ \simeq \frac{M}{1 - \varepsilon \cos(\omega_0 \varphi + \varphi_0)},
$$

can be determined in a simple way for $\lambda = 0$.
We have to find solutions of a third order algebraic equation in order to determine possible horizons for nontrivial λ.

We generate families of 3–d solitonic wave equation of Kadomtev–Petviashvili (KP) type if the generating function is any $h_4(\tau, x^1, y^3) = h(\tau, x^1, y^3)$ as a solution of

$$\pm\partial_{11}^2 h + (\partial_\tau h + hh^* + \epsilon h^{***})^* = 0.$$

The dispersionless limit is characterized by $\epsilon \to 0$ and Burgers' eq $\partial_\tau h + hh^* = 0$.

Integrating above equation on y^3, we obtain $\partial_\tau h_4 = -h_4 h_4^* - \epsilon h_4^{***} \mp \int dy^3 \partial_{11}^2 h_4$.

Effective solitonic source $\Upsilon = \Lambda_0 - \ {}^{\phi}\Lambda - 2\varsigma^2 - hh^* - \epsilon h^{***} \mp \int dy^3 \partial_{11}^2 h$.

For a solution $h_4(\tau, x^1, y^3)$, compute $\Psi^2 = B(\tau, x^1) - \frac{4}{\Lambda_0} h_4$ and $h_3 = -\frac{(h_4^*)^2}{h_4[B(\tau,x^1) - \frac{4}{\Lambda_0} h_4]}$

for an integration function $B(\tau, x^1)$ (for simplicity) $h_3 = h_4 = h$

3-d KdP solitonic quadratic element

$$ds^2_{KdP} = g_{\alpha\beta}(\tau, x^1, y^3) du^\alpha du^\beta = e^{\psi(\tau, x^k)}[(dx^1)^2 + (dx^2)^2] + h(\tau, x^1, y^3)$$
$$\{\ [dy^3 + \frac{\partial_i(-\frac{\Lambda_0}{4} B(\tau,x^1) + h)}{h^*} dx^1]^2 + [dt + (\ {}_1n_k(x^1) + \ {}_2\widetilde{n}_k(x^1) \int dy^3 \sqrt{|h|})^{-1}) dx^1]^2\}.$$

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Conclusions

1. The AFDM is a geometric method for decoupling and integrating system of nonlinear PDEs for general ansatz not reducing (modified) Einstein eqs to ODEs. Using nonolonomic frame transforms and distortions of connections, we construct generic off–diagonal solutions in MGTs depending on all spacetime coordinates via generating and integration functions and various parameters: In N–adapted form: the Cauchy problem, asymptotic / boundary conditions, various topology and symmetry configurations; smooth conditions and modelling stochastic, fractional and noncommutative interactions.
2. Modeling MGTs and OFF-DIAGONAL SOLUTIONS: effective massive terms and/or two non-Riemannian volume forms; associated bimetric and/or biconnection geometric structures; modified Lagrangians $f(R, T)$, nonholonomically induced torsion and nonholonomic constraints for LC-configurations.
3. INTERPRETATION:
   – running and polarization constants (small parameters and general ones)
   – deformation of horizons (ellipsoidal ones) and topological transitions (torus)
   – black ellipsoids / toroids; anisotropic cosmological metrics, wormholes, Taub NUT; Lie algebroid structures; solitonic gravitational hierarchies
   – spinor and almost symplectic structures; noncommutative deformations