



21st International Conference  
on General Relativity  
and Gravitation  
Columbia University, New York

# Accelerating CBC parameter estimation with multi-band frequency domain waveforms

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*Supervisors: Ilya Mandel, John Veitch*

GraWIToN



GW Initial Training Network

UNIVERSITY OF  
BIRMINGHAM



What?

why?

when?

which?

what?



*I'm going to tell you...*

What is the problem, why it  
is important to solve it  
soon, which solution we  
propose and what are the  
results already achieved.



# COMPACT BINARY COALESCENCES

What?

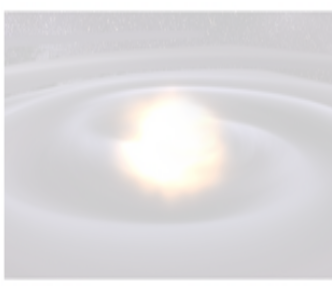
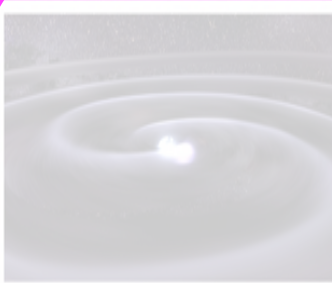
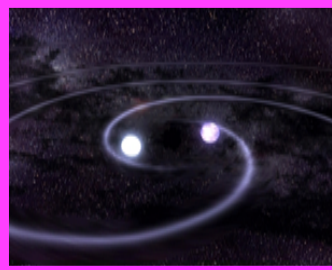
Why?

When?

Which?

What?

- Neutron Star-Neutron Star [NS-NS];
- Neutron Star-Black Hole [NS-BH];
- Black Hole-Black Hole [BH-BH]





What?

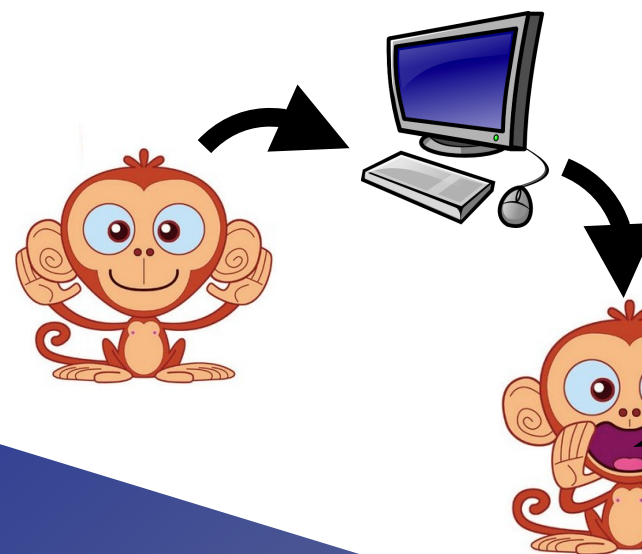
Why?

When?

Which?

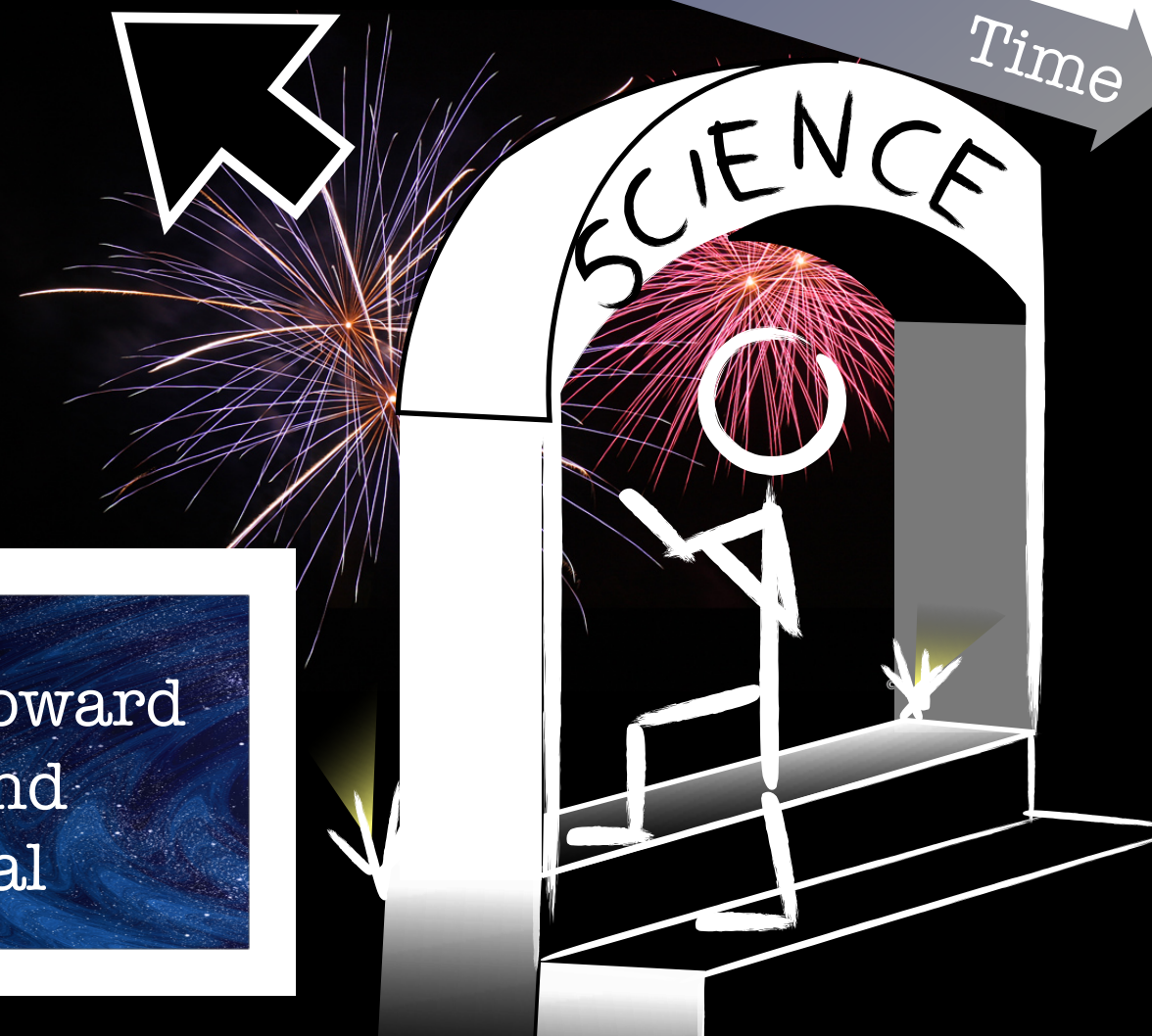
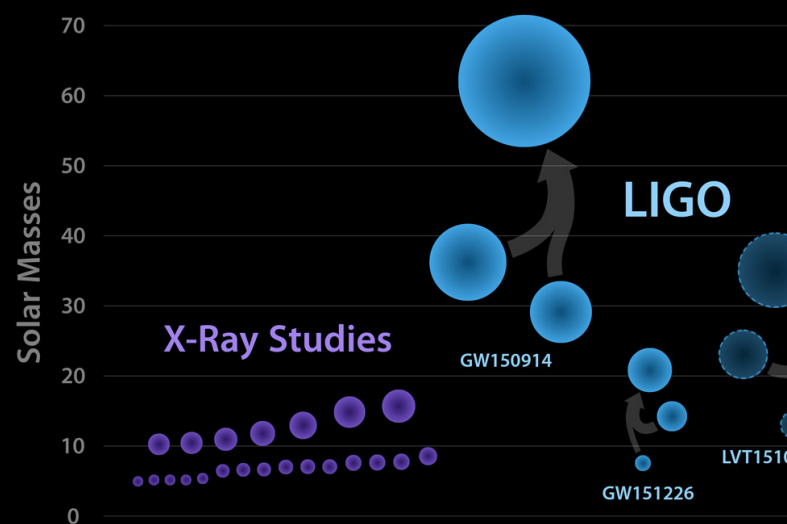
What?

# PARAMETER ESTIMATION



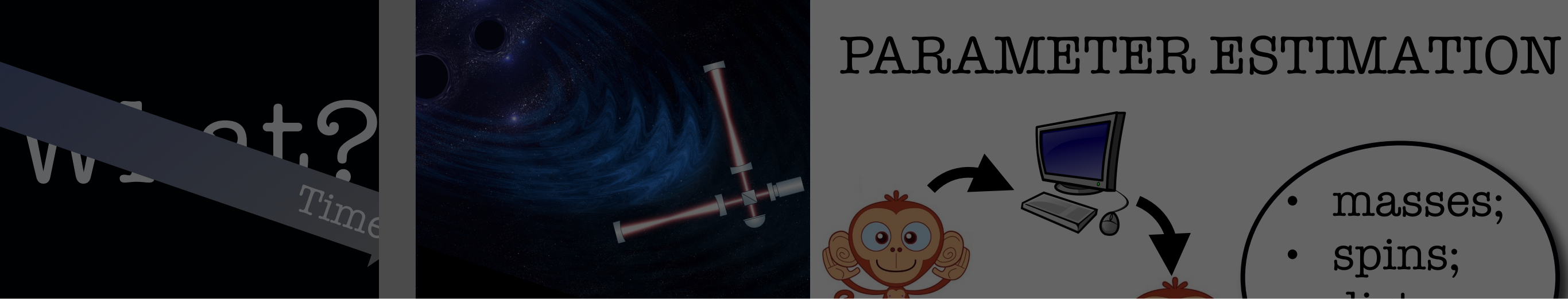
- masses;
- spins;
- distance;
- ..

Black Holes of Known Mass



PE is our **first step** toward  
GW astronomy and  
science in general





# PE IS CRUCIAL





What?

**PE IS BASED ON THE BAYES' THEOREM**

$\mathbf{d}$  : data  
 $H$  : model

$$\mathbf{d} = \mathbf{h}(\underset{\substack{\uparrow \\ \text{parameters}}}{\boldsymbol{\theta}}) + \mathbf{n}$$

$\mathbf{h}$  : signal  
 $\mathbf{n}$  : noise

Why?

When?

Which?

What?

$$p(\boldsymbol{\theta} | \mathbf{d}, H) = \frac{p(\boldsymbol{\theta} | H) p(\mathbf{d} | \boldsymbol{\theta}, H)}{p(\mathbf{d} | H)}$$



What?

# PE IS BASED ON THE BAYES' THEOREM

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 $H$  : model

$$\mathbf{d} = \mathbf{h}(\boldsymbol{\theta}) + \mathbf{n}$$

↑  
parameters

$\mathbf{h}$  : signal  
 $\mathbf{n}$  : noise

Why?

$$p(\boldsymbol{\theta}|\mathbf{d}, H) = \frac{p(\boldsymbol{\theta}|H)p(\mathbf{d}|\boldsymbol{\theta}, H)}{p(\mathbf{d}|H)}$$

Posterior

When?

Which?

What?



What?

# PE IS BASED ON THE BAYES' THEOREM

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parameters

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Why?

$$p(\boldsymbol{\theta}|\mathbf{d}, H) = \frac{p(\boldsymbol{\theta}|H)p(\mathbf{d}|\boldsymbol{\theta}, H)}{p(\mathbf{d}|H)}$$

Priors

Posterior

When?

Which?

What?



What?

# PE IS BASED ON THE BAYES' THEOREM

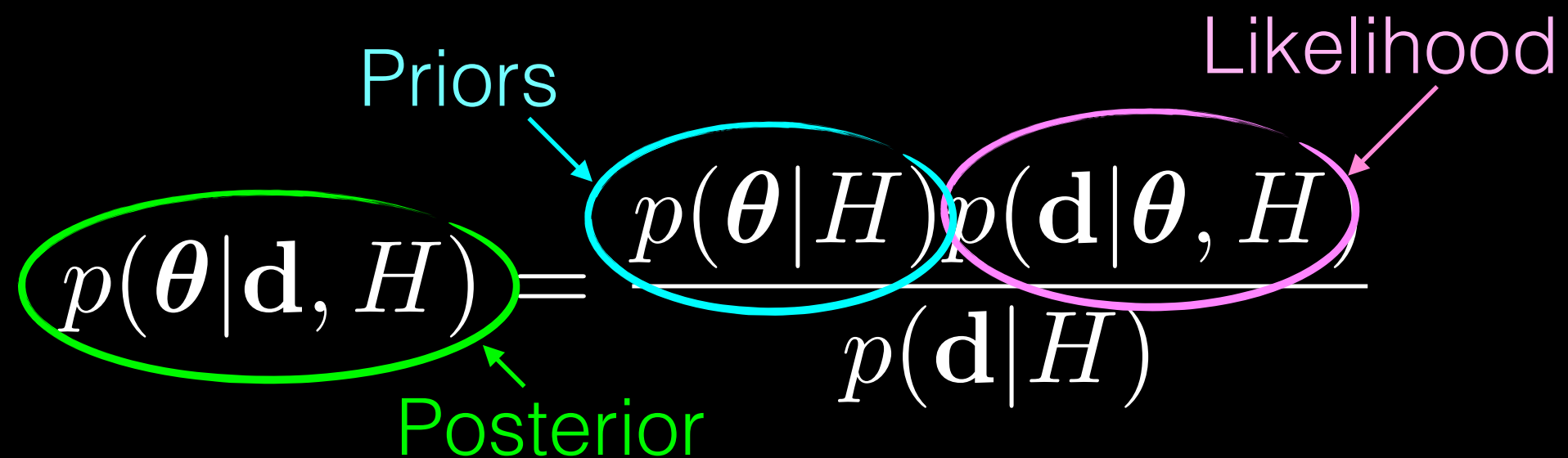
$\mathbf{d}$  : data  
 $H$  : model

$$\mathbf{d} = \mathbf{h}(\boldsymbol{\theta}) + \mathbf{n}$$

↑  
parameters

$\mathbf{h}$  : signal  
 $\mathbf{n}$  : noise

Why?



The diagram illustrates Bayes' Theorem with the following components:

- Posterior:**  $p(\boldsymbol{\theta}|\mathbf{d}, H)$  (circled in green)
- Priors:**  $p(\boldsymbol{\theta}|H)$  (circled in cyan)
- Likelihood:**  $p(\mathbf{d}|\boldsymbol{\theta}, H)$  (circled in magenta)

$$p(\boldsymbol{\theta}|\mathbf{d}, H) = \frac{p(\boldsymbol{\theta}|H)p(\mathbf{d}|\boldsymbol{\theta}, H)}{p(\mathbf{d}|H)}$$

When?

Which?

What?



What?

# PE IS BASED ON THE BAYES' THEOREM

$\mathbf{d}$  : data  
 $H$  : model

$$\mathbf{d} = \mathbf{h}(\boldsymbol{\theta}) + \mathbf{n}$$

↑  
parameters

$\mathbf{h}$  : signal  
 $\mathbf{n}$  : noise

Why?

When?

Which?

What?

The diagram illustrates Bayes' Theorem with the following components:

- Priors**: A cyan arrow points to the term  $p(\boldsymbol{\theta}|H)$ , which is enclosed in a cyan ellipse.
- Likelihood**: A magenta arrow points to the term  $p(\mathbf{d}|\boldsymbol{\theta}, H)$ , which is enclosed in a magenta ellipse.
- Evidence**: A yellow arrow points to the term  $p(\mathbf{d}|H)$ , which is enclosed in a yellow ellipse.
- Posterior**: A green arrow points to the term  $p(\boldsymbol{\theta}|\mathbf{d}, H)$ , which is enclosed in a green ellipse.

The equation is presented as:

$$p(\boldsymbol{\theta}|\mathbf{d}, H) = \frac{p(\boldsymbol{\theta}|H)p(\mathbf{d}|\boldsymbol{\theta}, H)}{p(\mathbf{d}|H)}$$



What?

Why?

When?

Which?

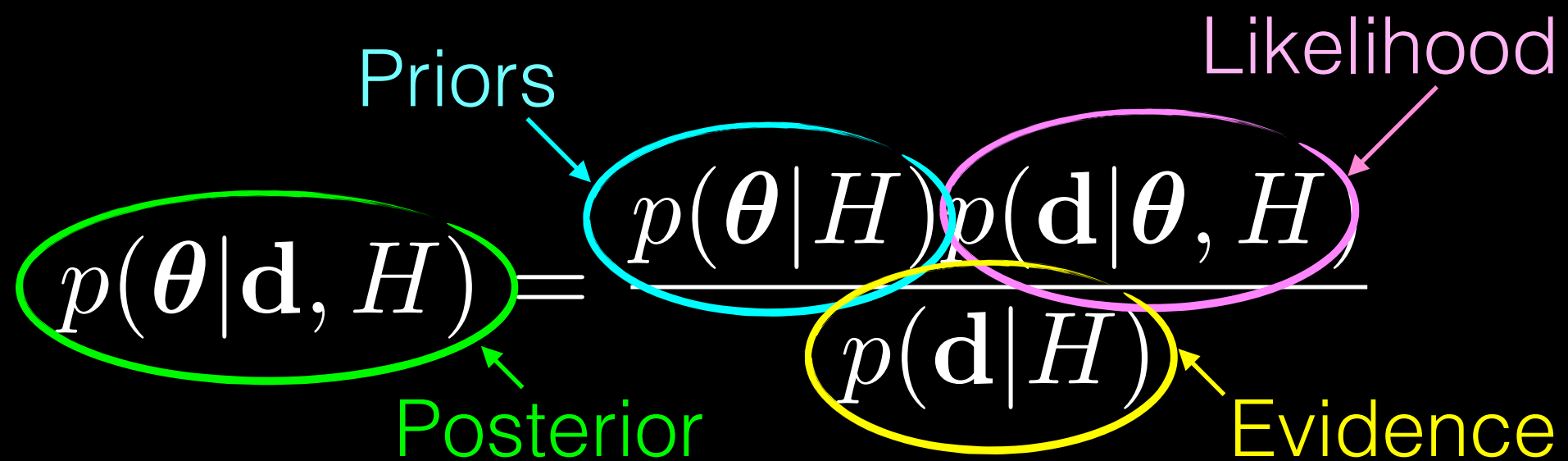
What?

## PE IS BASED ON THE BAYES' THEOREM

$\mathbf{d}$  : data  
 $H$  : model

$$\mathbf{d} = \underset{\substack{\uparrow \\ \text{parameters}}}{\mathbf{h}(\boldsymbol{\theta})} + \mathbf{n}$$

$\mathbf{h}$  : signal  
 $\mathbf{n}$  : noise


$$\underbrace{p(\boldsymbol{\theta}|\mathbf{d}, H)}_{\text{Posterior}} = \frac{\underbrace{p(\boldsymbol{\theta}|H)}_{\text{Priors}} \underbrace{p(\mathbf{d}|\boldsymbol{\theta}, H)}_{\text{Likelihood}}}{\underbrace{p(\mathbf{d}|H)}_{\text{Evidence}}}$$

## GAUSSIAN NOISE

- zero mean
- known variance

$$p(\mathbf{d}|\boldsymbol{\theta}, H) \propto \exp \left[ -\delta f \sum_{i=0}^N \frac{2|\tilde{d}(f_i) - \tilde{h}(\boldsymbol{\theta}, f_i)|^2}{S_n(f_i)} \right]$$

What?

Why?

When?

Which?

What?

# PE IS BASED ON THE BAYES' THEOREM

$\mathbf{d}$  : data  
 $H$  : model

$$\mathbf{d} = \underset{\substack{\uparrow \\ \text{parameters}}}{\mathbf{h}(\boldsymbol{\theta})} + \mathbf{n}$$

$\mathbf{h}$  : signal  
 $\mathbf{n}$  : noise

Diagram illustrating Bayes' theorem with colored ovals and arrows:

- Priors** (cyan oval) points to  $p(\boldsymbol{\theta}|H)$
- Likelihood** (purple oval) points to  $p(\mathbf{d}|\boldsymbol{\theta}, H)$
- Evidence** (yellow oval) points to  $p(\mathbf{d}|H)$
- Posterior** (green oval) points to  $p(\boldsymbol{\theta}|\mathbf{d}, H)$

$$p(\boldsymbol{\theta}|\mathbf{d}, H) = \frac{p(\boldsymbol{\theta}|H) p(\mathbf{d}|\boldsymbol{\theta}, H)}{p(\mathbf{d}|H)}$$

## GAUSSIAN NOISE

- zero mean
- known variance

Equation for Gaussian noise likelihood with red ovals and arrows:

- Red oval around  $N$  in the summation, with an arrow from "zero mean".
- Red oval around  $|\tilde{\mathbf{h}}(\boldsymbol{\theta}, f_i)|^2$ , with an arrow from "known variance".

$$p(\mathbf{d}|\boldsymbol{\theta}, H) \propto \exp \left[ -\delta f \sum_{i=0}^N \frac{2|\tilde{\mathbf{d}}(f_i) - \tilde{\mathbf{h}}(\boldsymbol{\theta}, f_i)|^2}{S_n(f_i)} \right]$$



What?

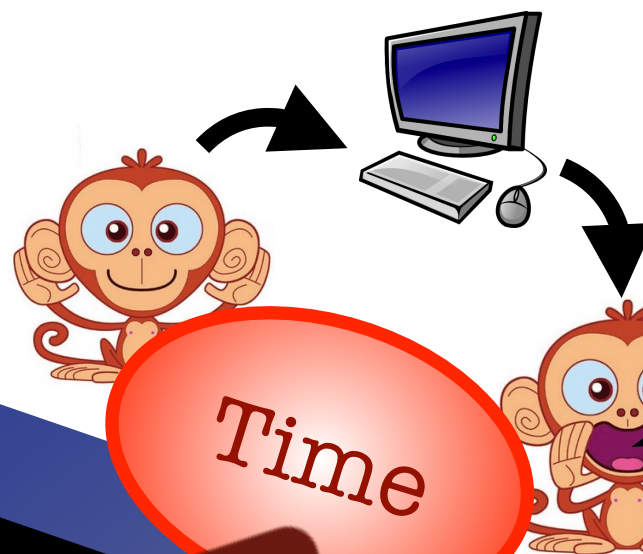
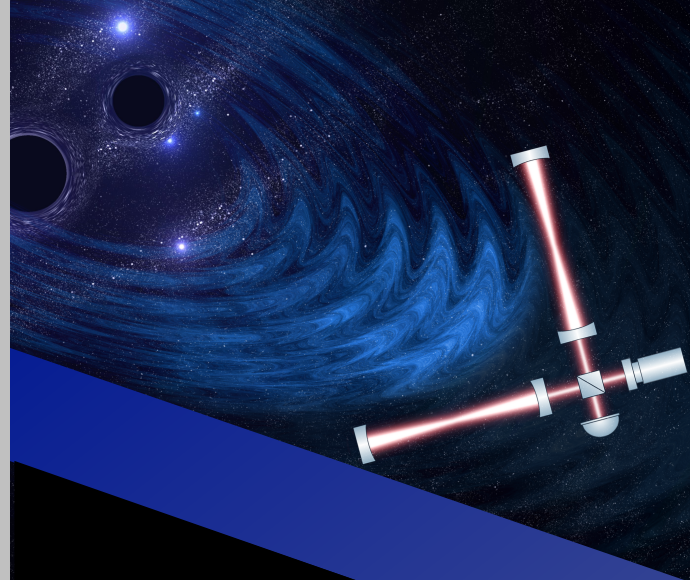
Why?

When?

Which?

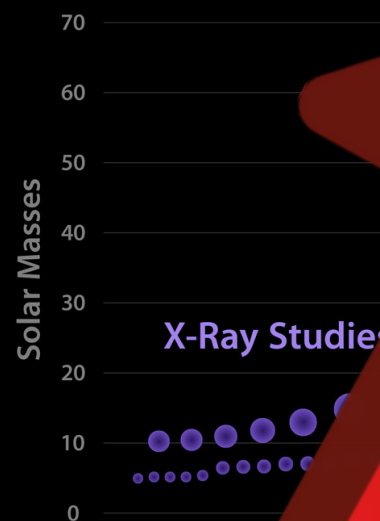
What?

# PARAMETER ESTIMATION



- masses;
- spins;
- distance;
- ..

Black Holes of Known Mass



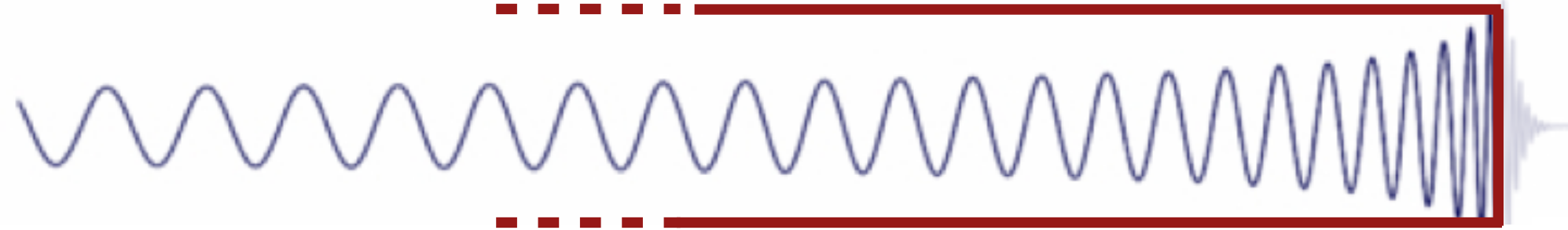
PARAMETER ESTIMATION

# PE IS EXPENSIVE

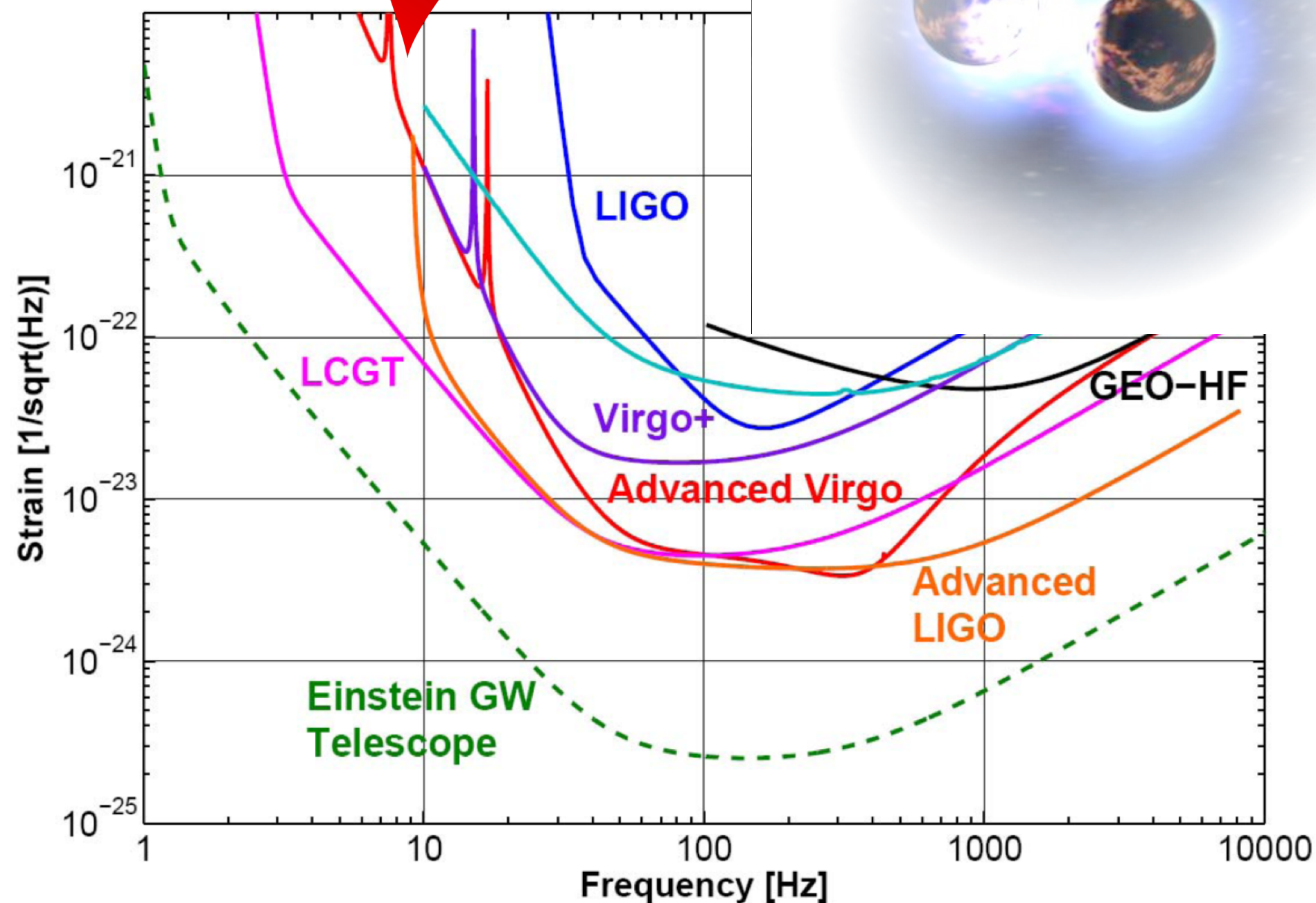
PE is moving toward  
GW astronomy and  
science in general



What?  
Why?  
When?  
Which?  
What?



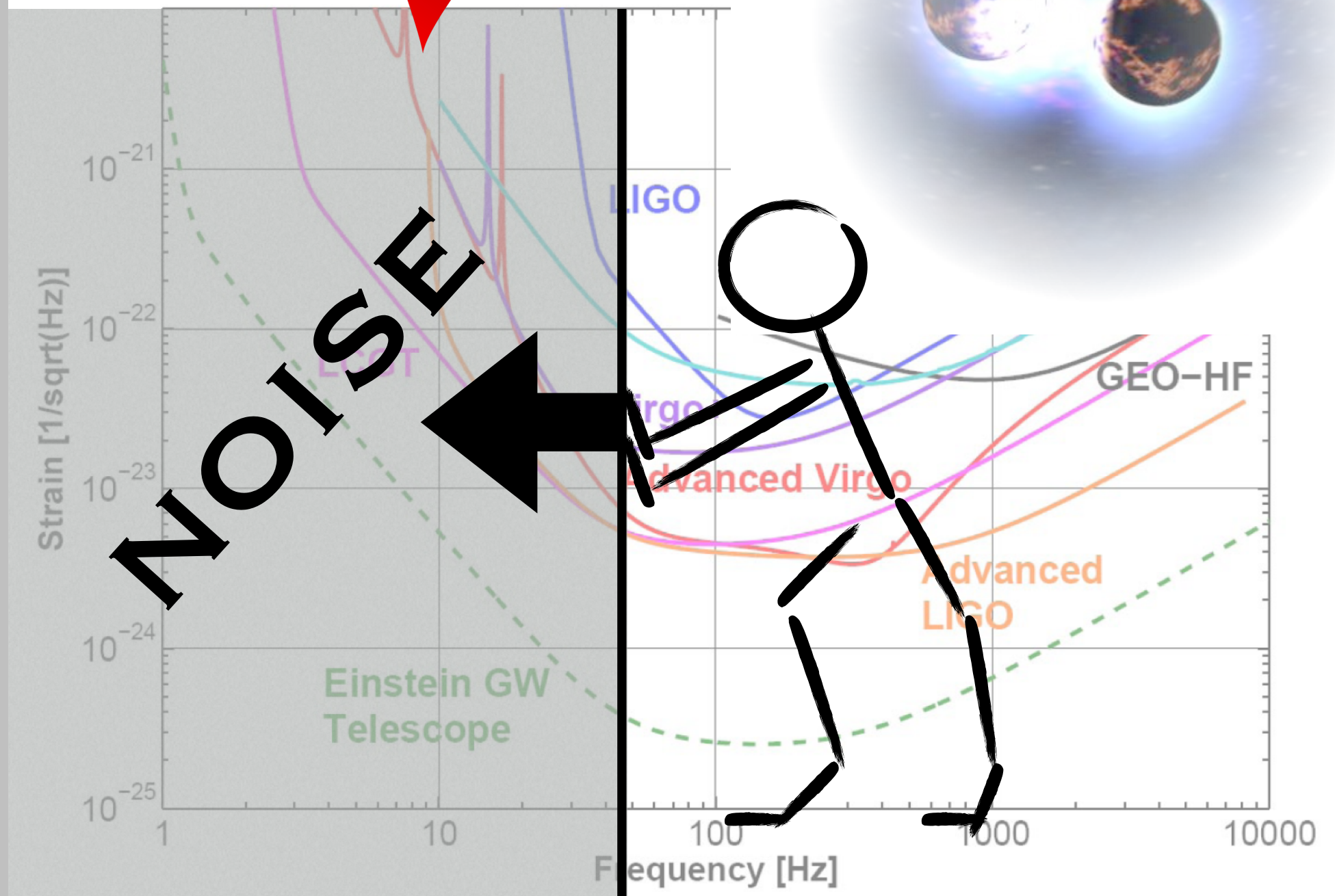
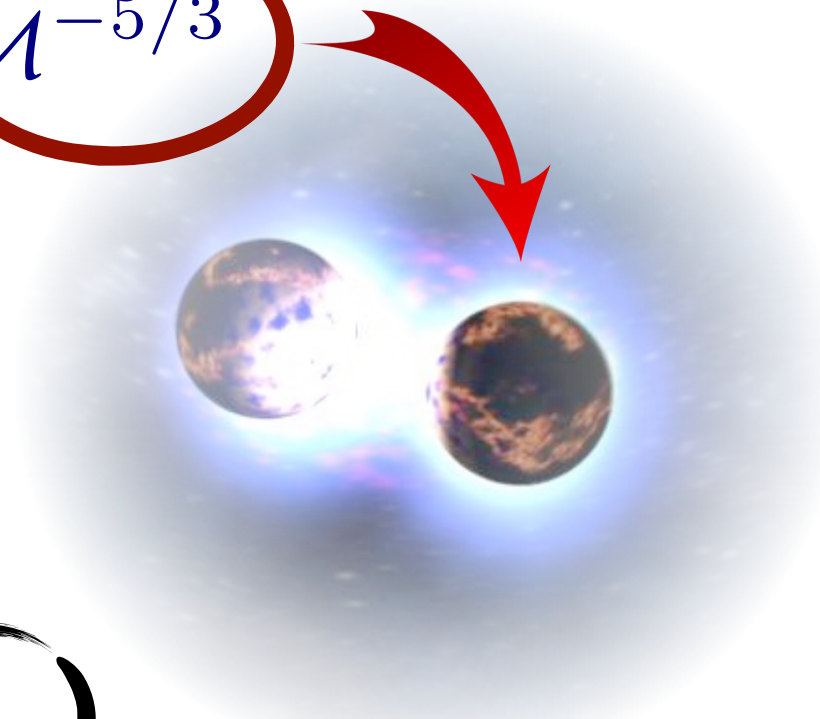
$$t_{run} \propto t_{inspiral} \propto f_{min}^{-8/3} M^{-5/3}$$



What?  
Why?  
When?  
Which?  
What?



$$t_{run} \propto t_{inspiral} \propto f_{min}^{-8/3} M^{-5/3}$$





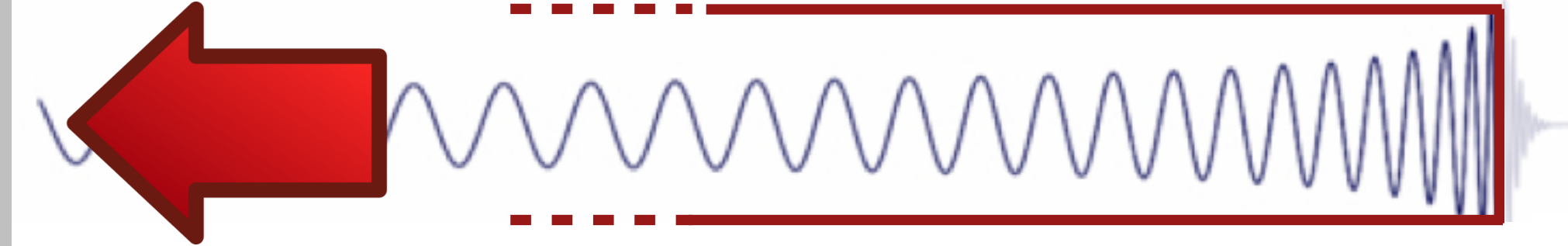
What?

Why?

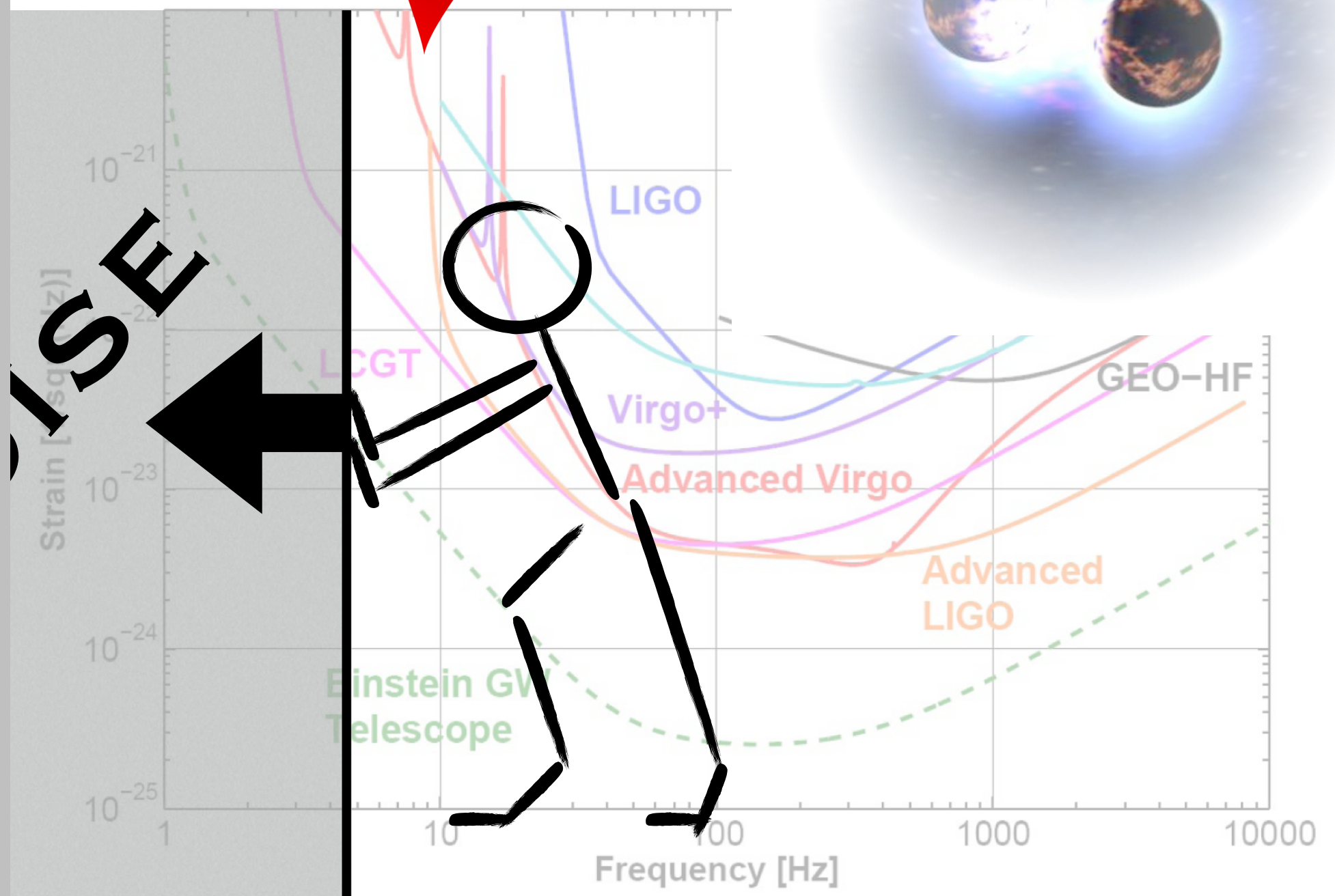
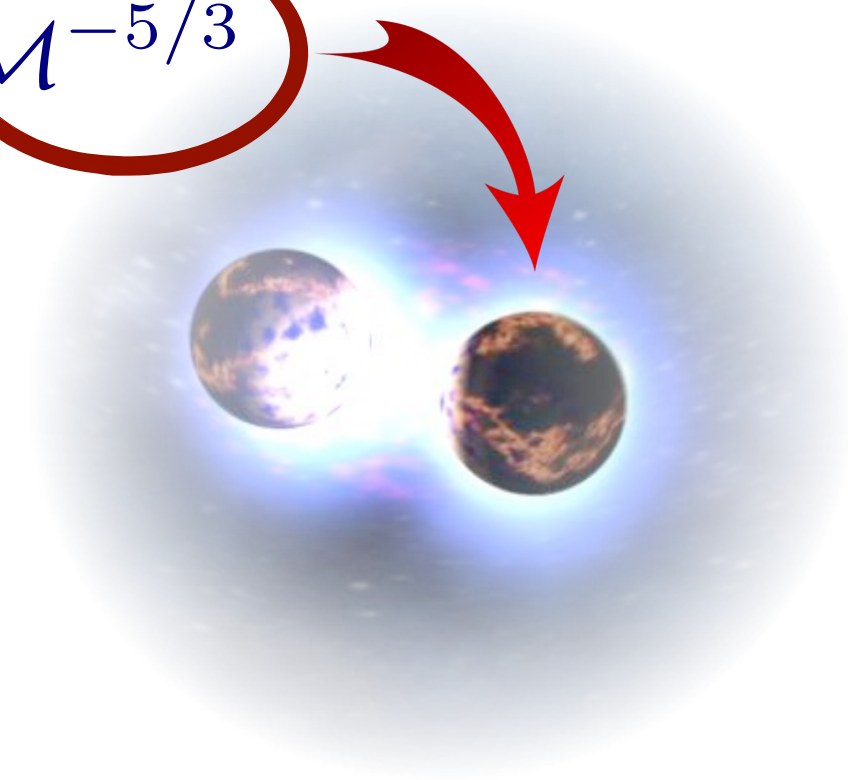
When?

Which?

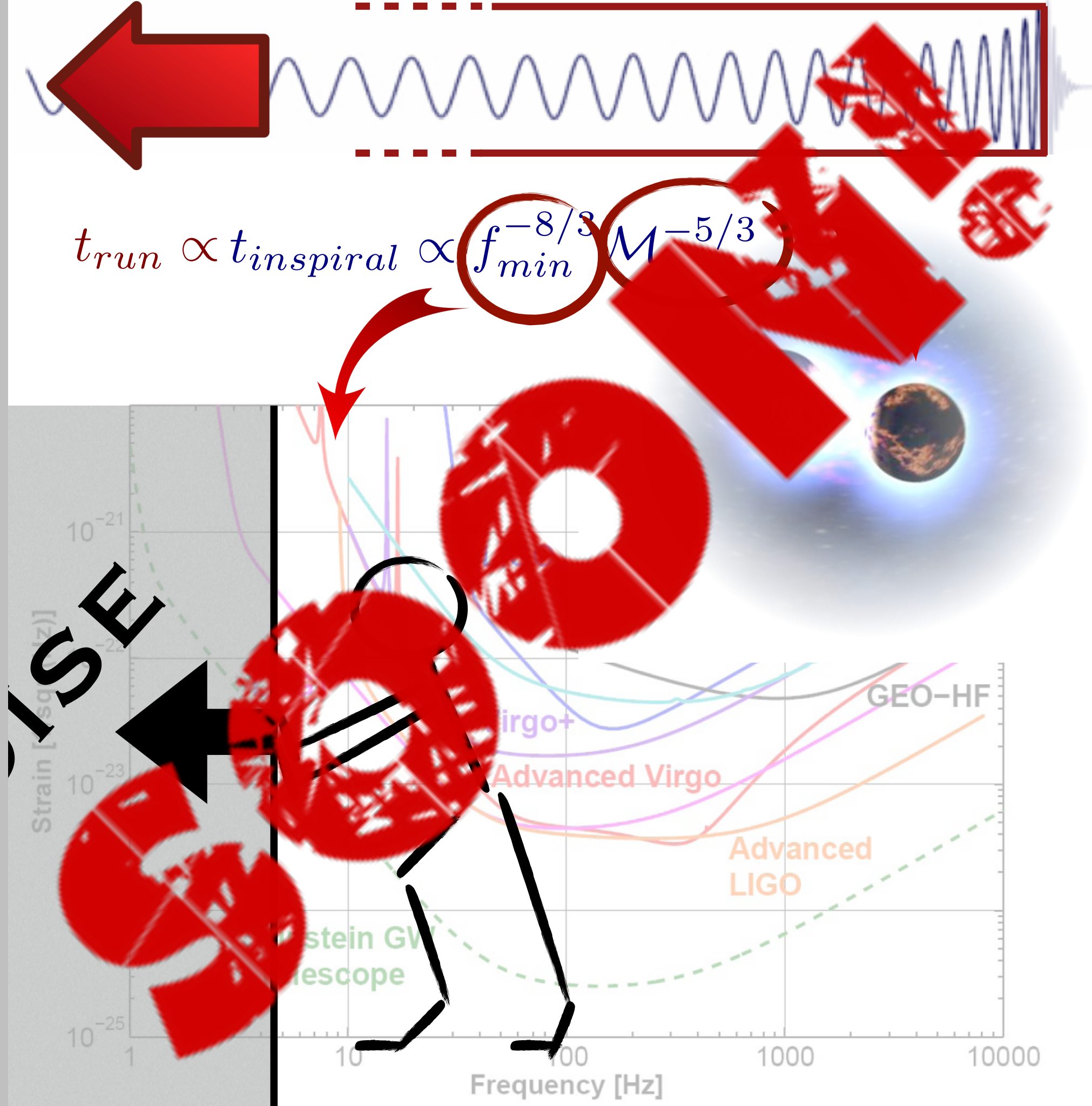
What?



$$t_{run} \propto t_{inspiral} \propto f_{min}^{-8/3} M^{-5/3}$$



What?  
Why?  
When?  
Which?  
What?





What?

Why?

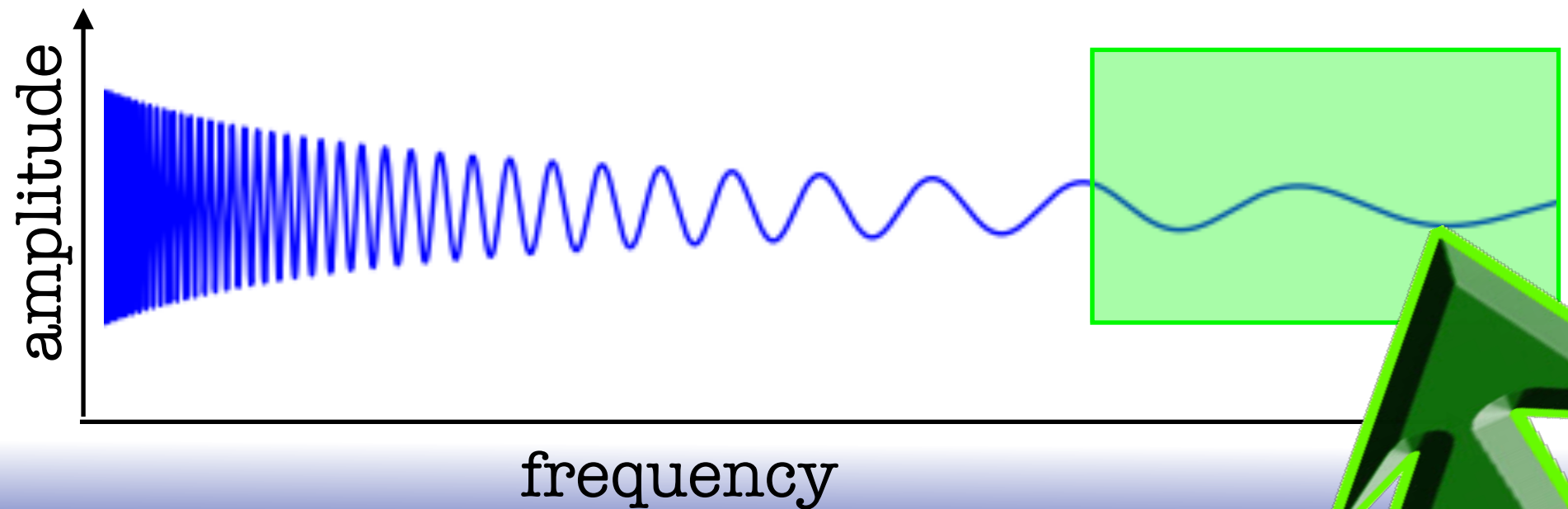
When?

Which?

What?

# FREQUENCY DOMAIN WAVEFORM

## THE CHIRPING FEATURE IN FREQUENCY DOMAIN



The latest parts of the waveform don't need the same sampling of the first ones

## SAMPLING THEORY

$$\delta f \leq T[s]^{-1}$$

~ONE POINT EVERY CYCLE

What?

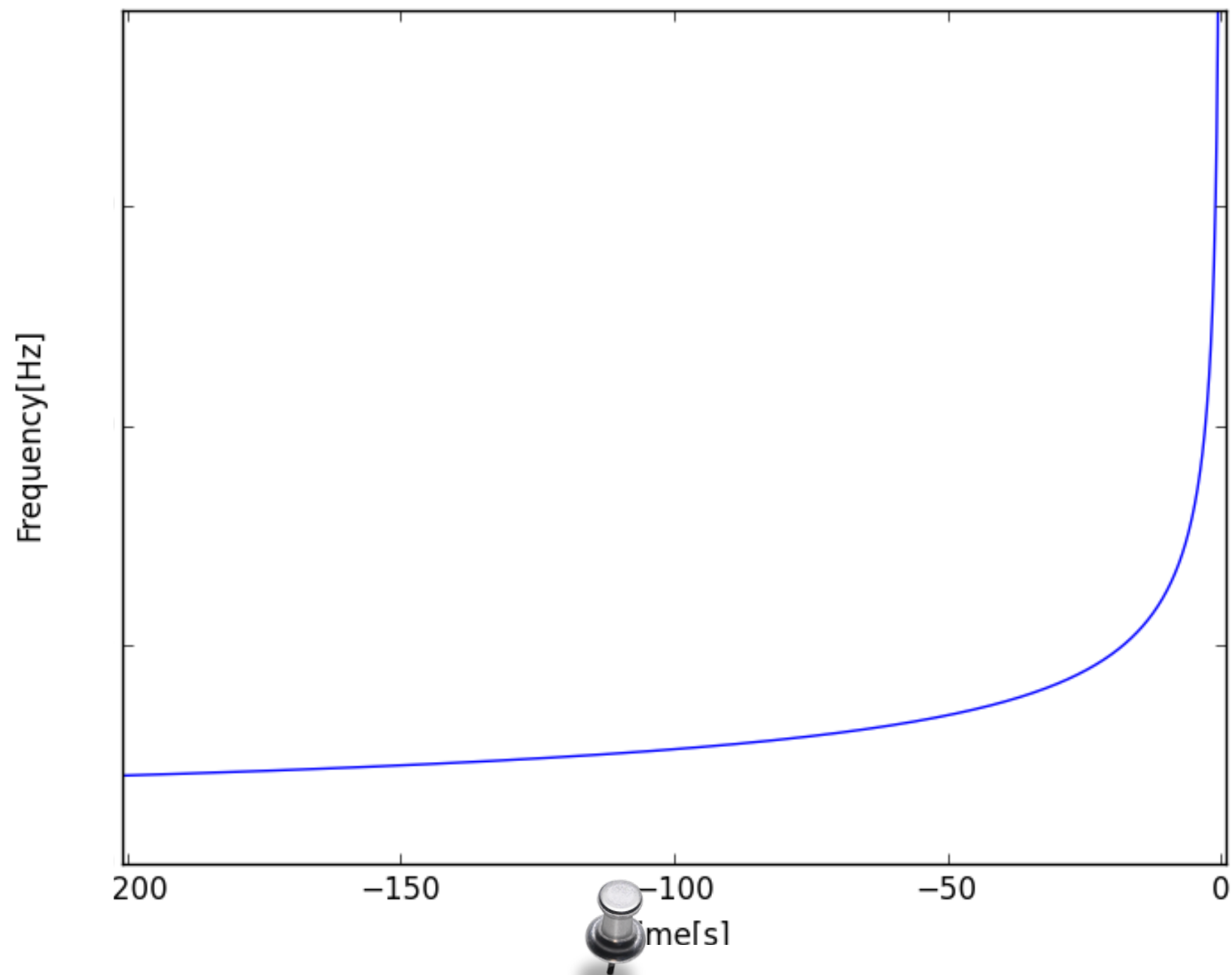
Why?

When?

Which?

What?

## FREQUENCY DOMAIN WAVEFORM



## SAMPLING THEORY

$$\delta f(t) \leq (T(t))^{-1}$$

WE CAN ADAPT THE SAMPLING FREQUENCY



What?

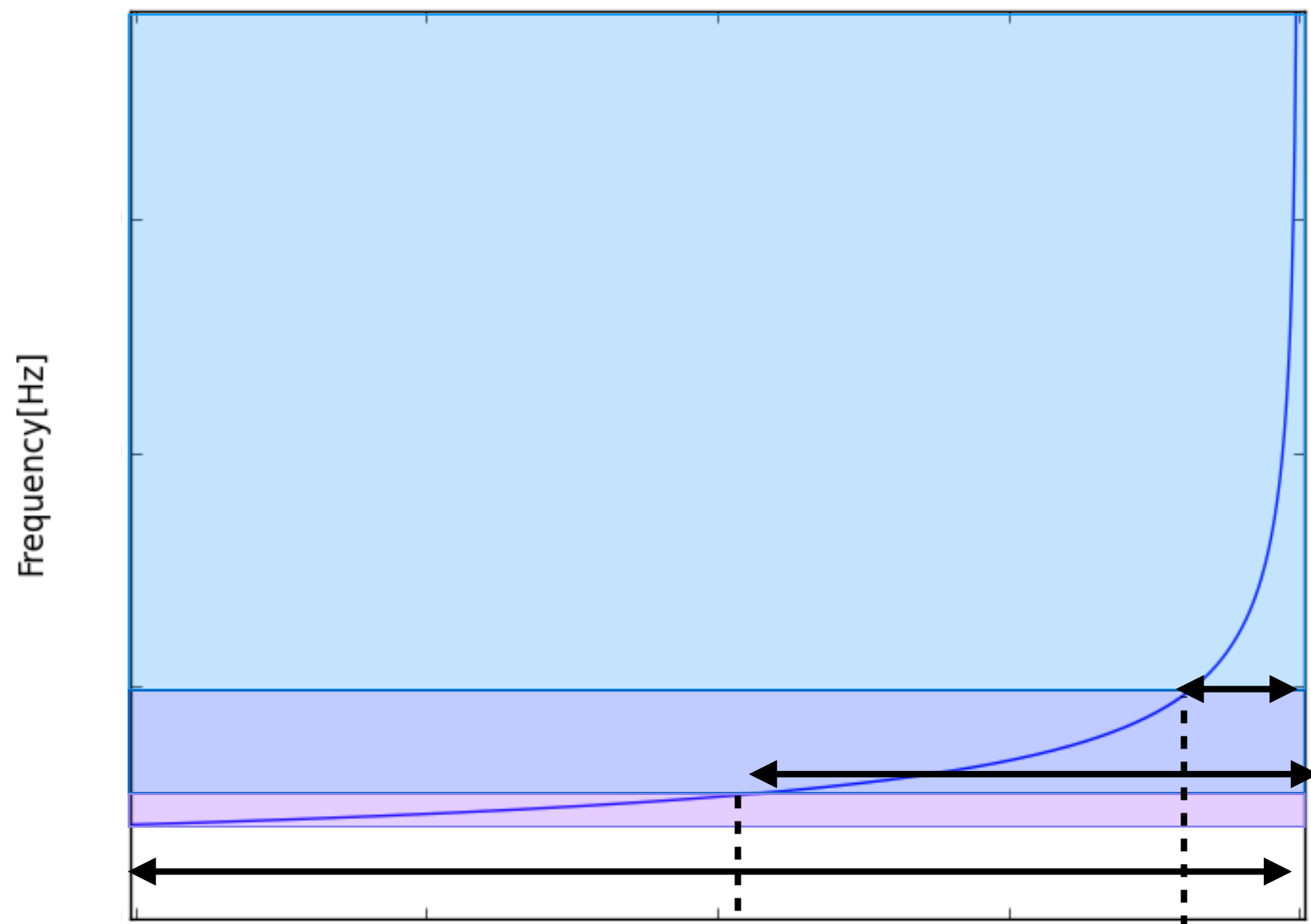
Why?

When?

Which?

What?

## FREQUENCY DOMAIN WAVEFORM



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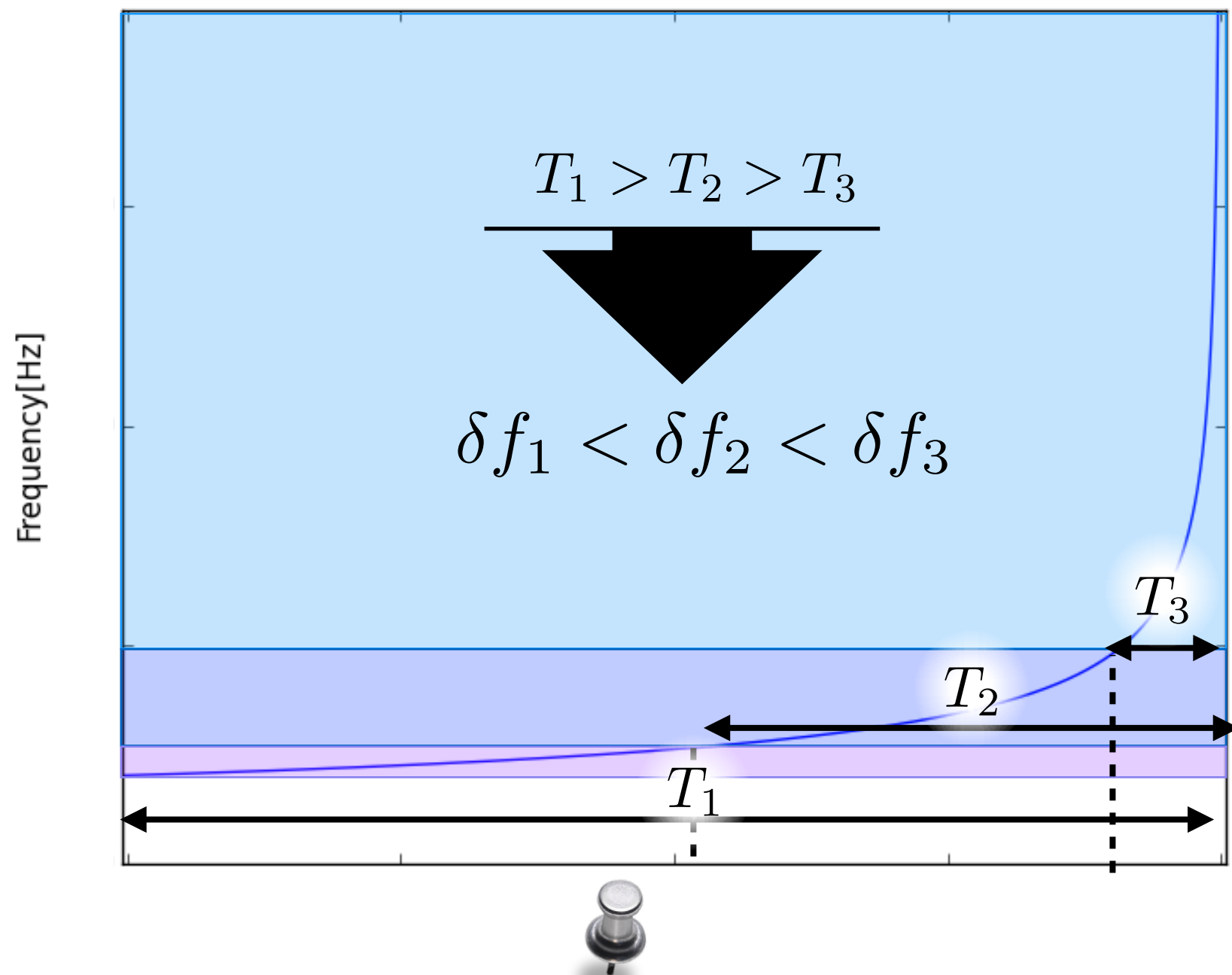
Why?

When?

Which?

What?

# FREQUENCY DOMAIN WAVEFORM



## SAMPLING THEORY

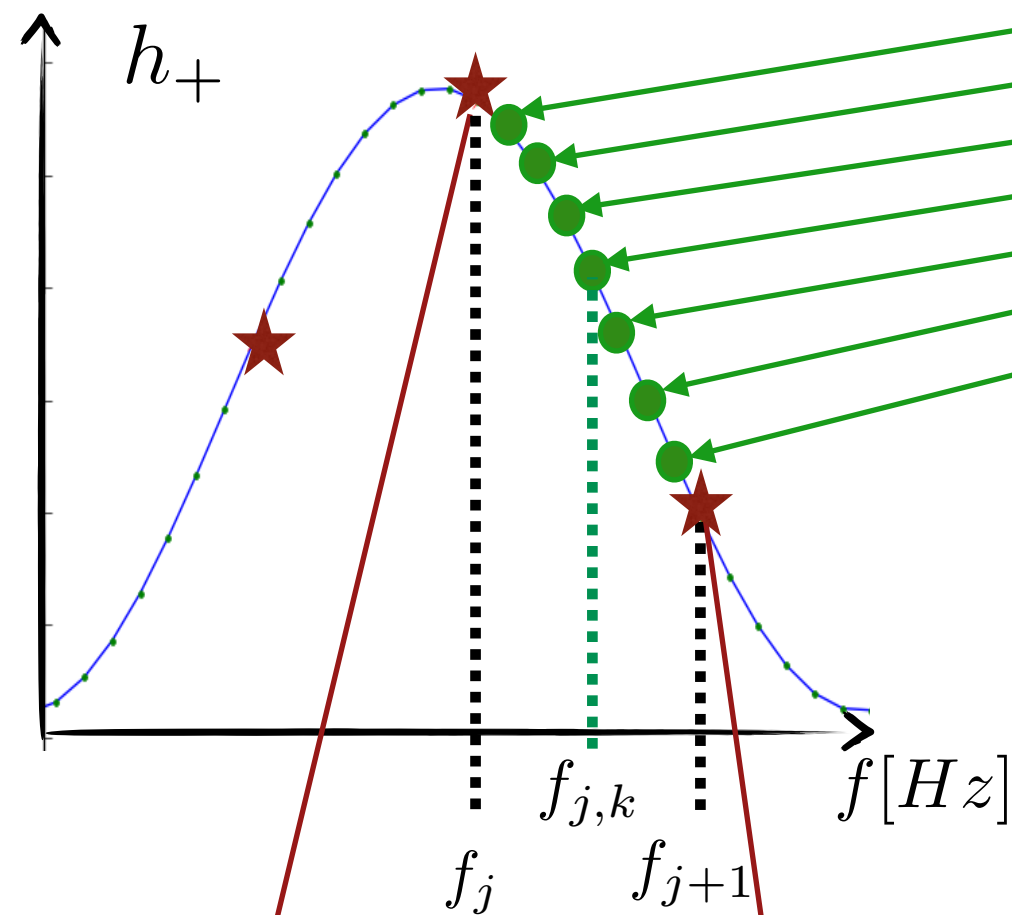
$$\delta f(t) \leq (T(t))^{-1}$$

WE CAN ADAPT THE SAMPLING FREQUENCY



# PHASE INTERPOLATION

$$\tilde{h}(\theta, f) \sim A(\theta, f)e^{i\psi(\theta, f)}$$



$$\tilde{h}_{jk} \approx \chi_{jk}$$

$$\chi_{jk} \equiv \tilde{h}(f_j)e^{i\Delta\psi_{jk}}$$

OUT

$$\psi(f) = \int_{f_j}^f \frac{d\psi(\hat{f})}{d\hat{f}} d\hat{f}$$

$$\psi(f_{jk})$$

$\cong$

$$\frac{\psi(f_{j+1}) - \psi(f_j)}{f_{j+1} - f_j} k \delta f_0$$

$$\tilde{h}(f_j), \tilde{h}(f_{j+1})$$

IN

What?

Why?

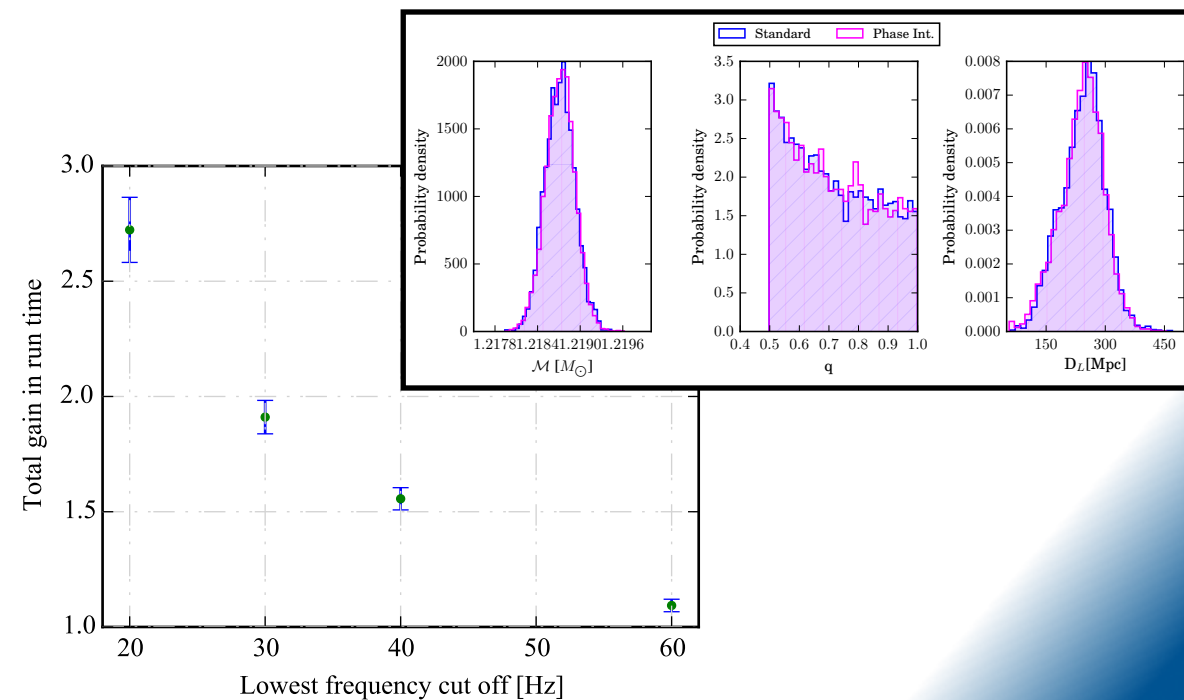
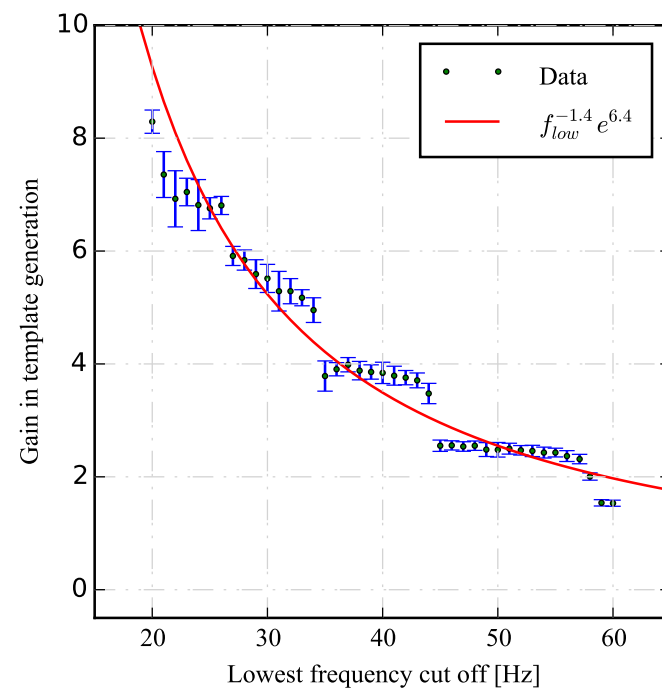
When?

Which?

What?

# RESULTS & MODEL DEPENDENCE

For one of the simplest waveform models : **TAYLORF2**





What?

Why?

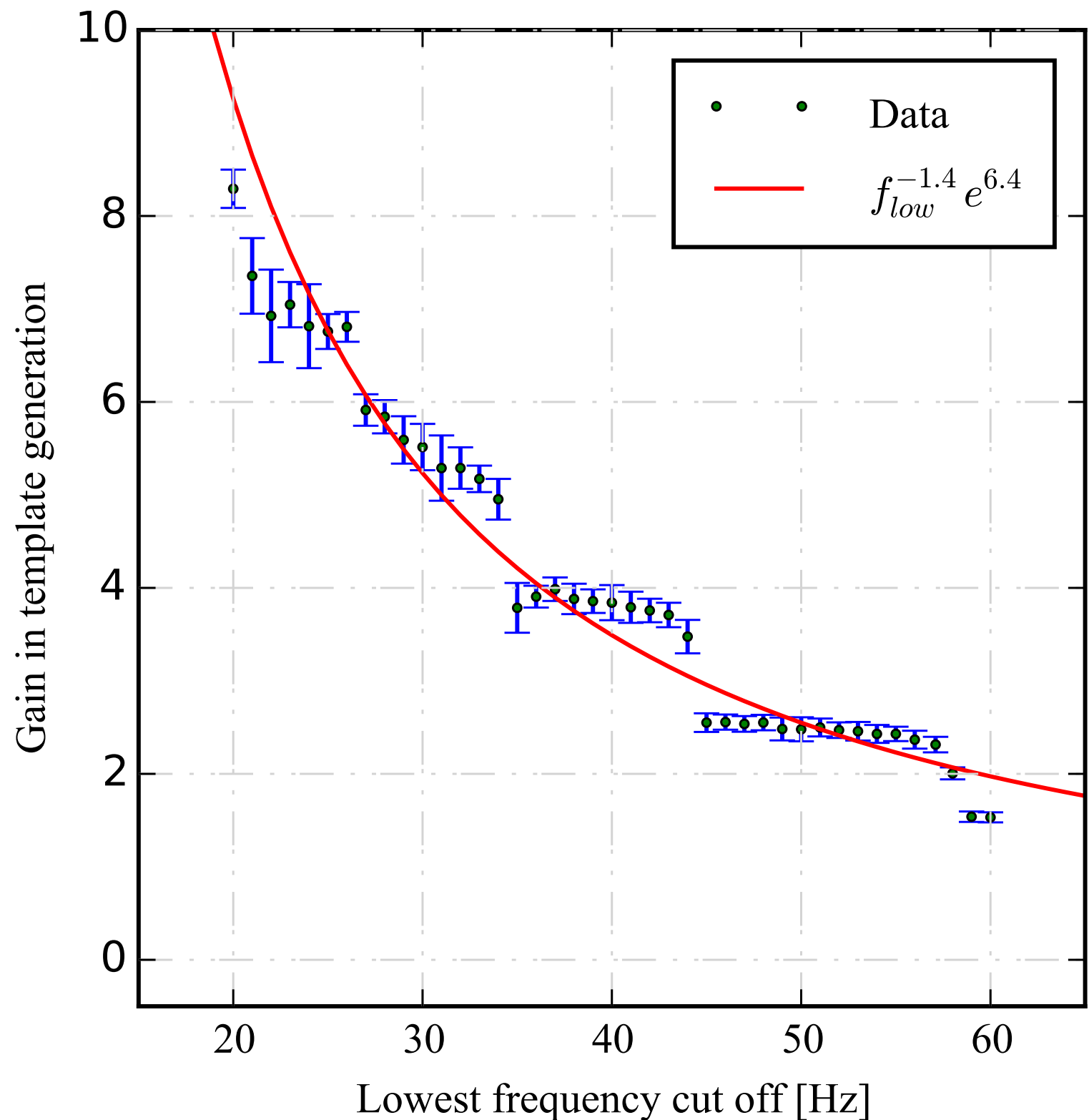
When?

Which?

What?

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For one of the simplest waveform models : **TAYLORF2**



What?

Why?

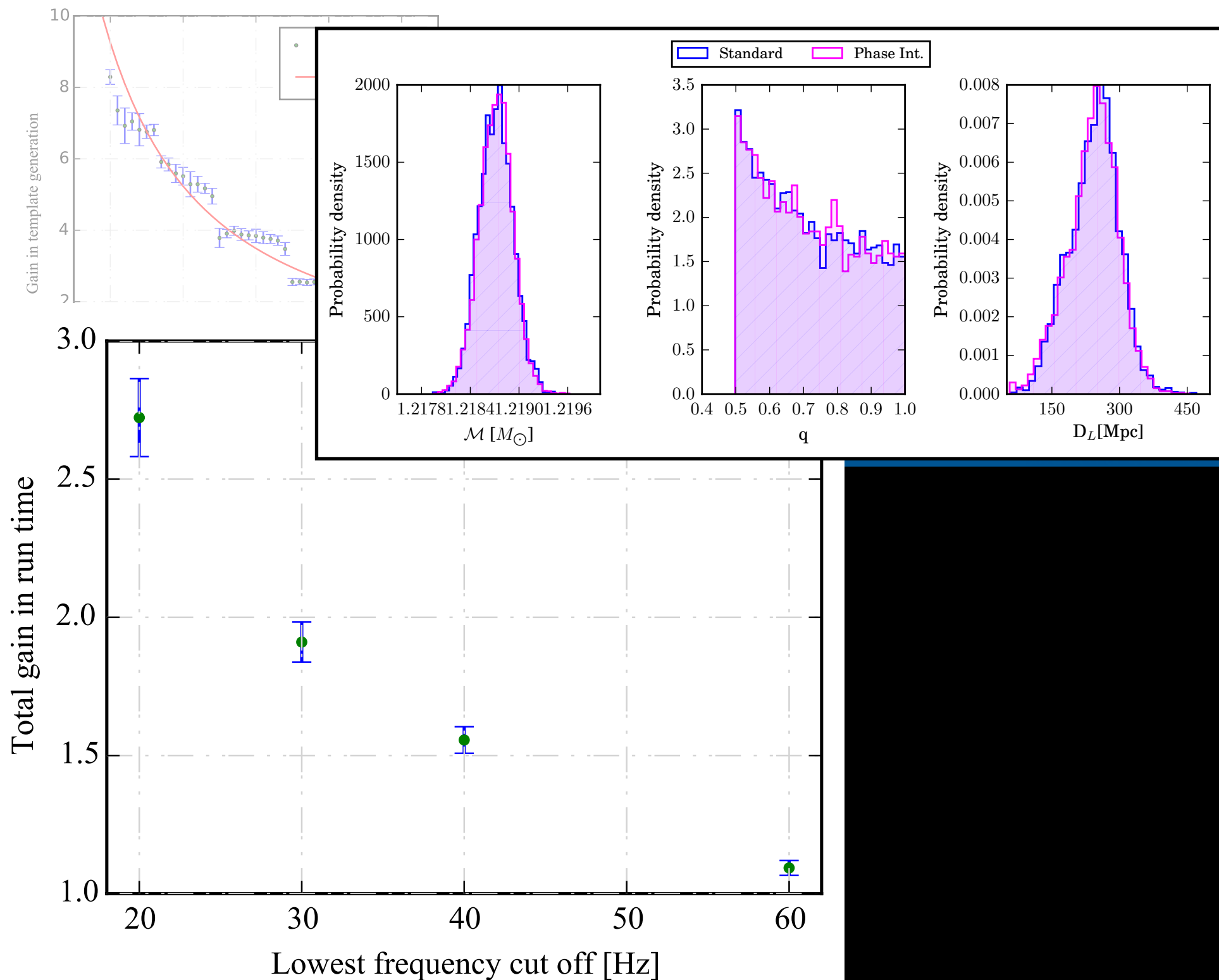
When?

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For one of the simplest waveform models : **TAYLORF2**





What?

Why?

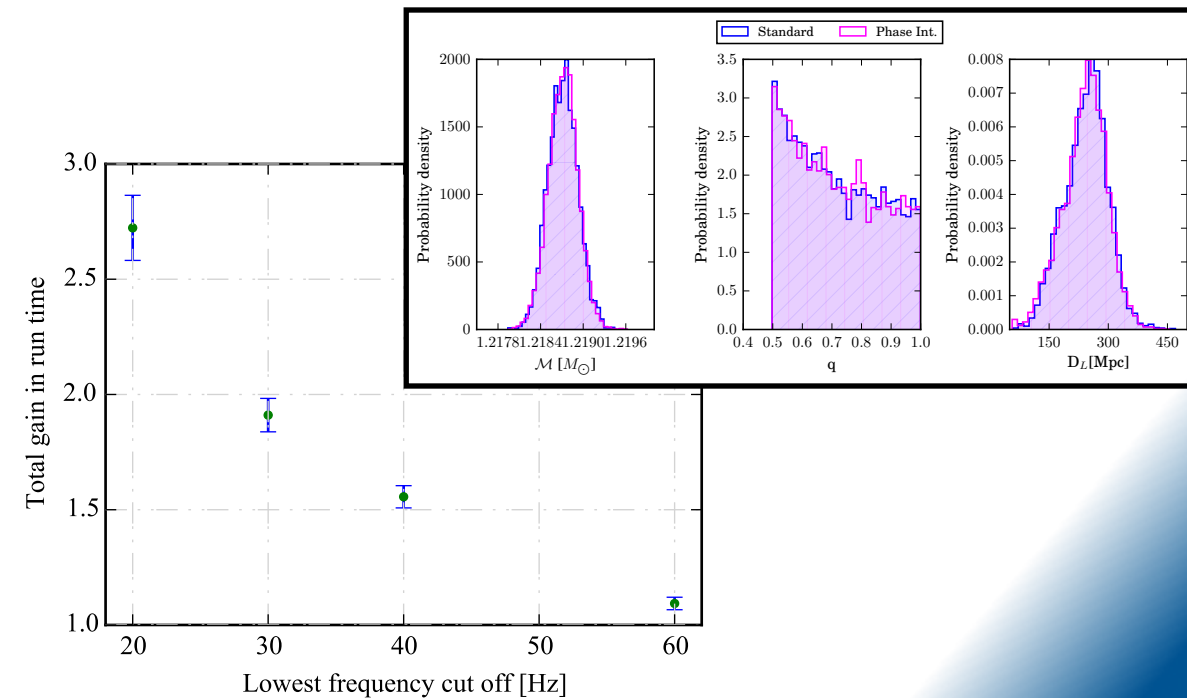
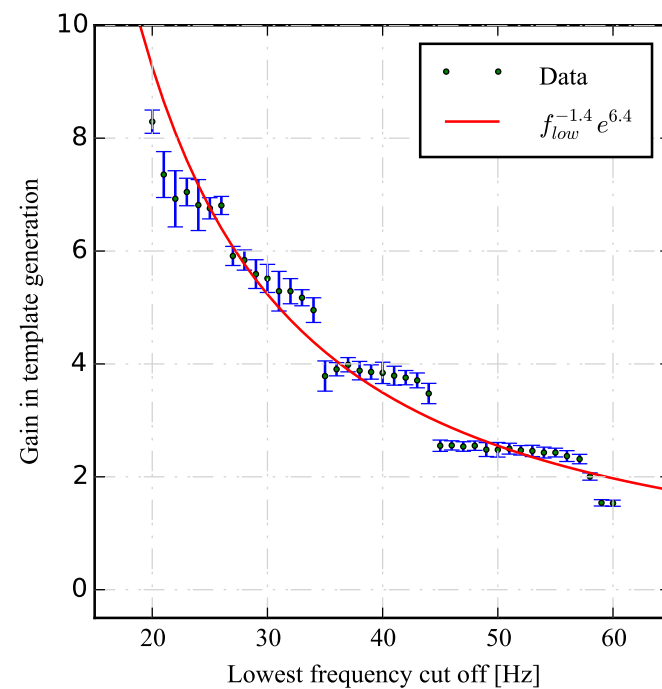
When?

Which?

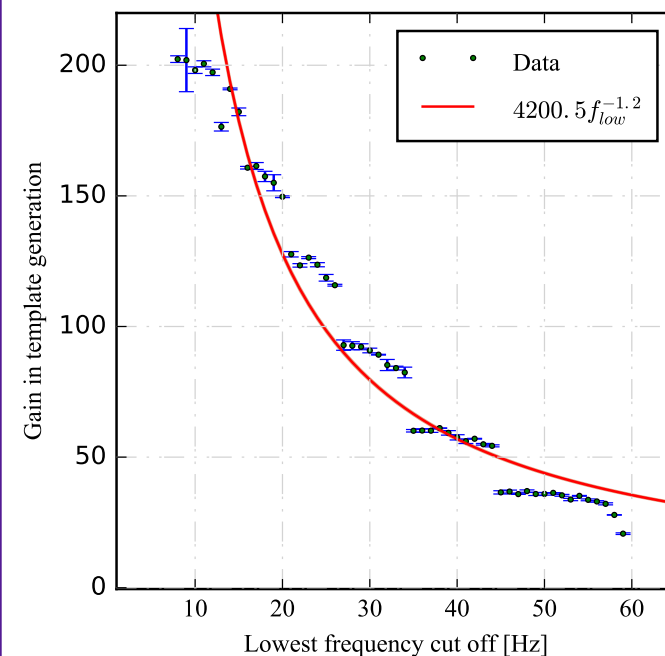
What?

# RESULTS & MODEL DEPENDENCE

For one of the simplest waveform models : **TAYLORF2**



For a more sophisticated waveform model :  
**IMRPHENOMPv2**



What?

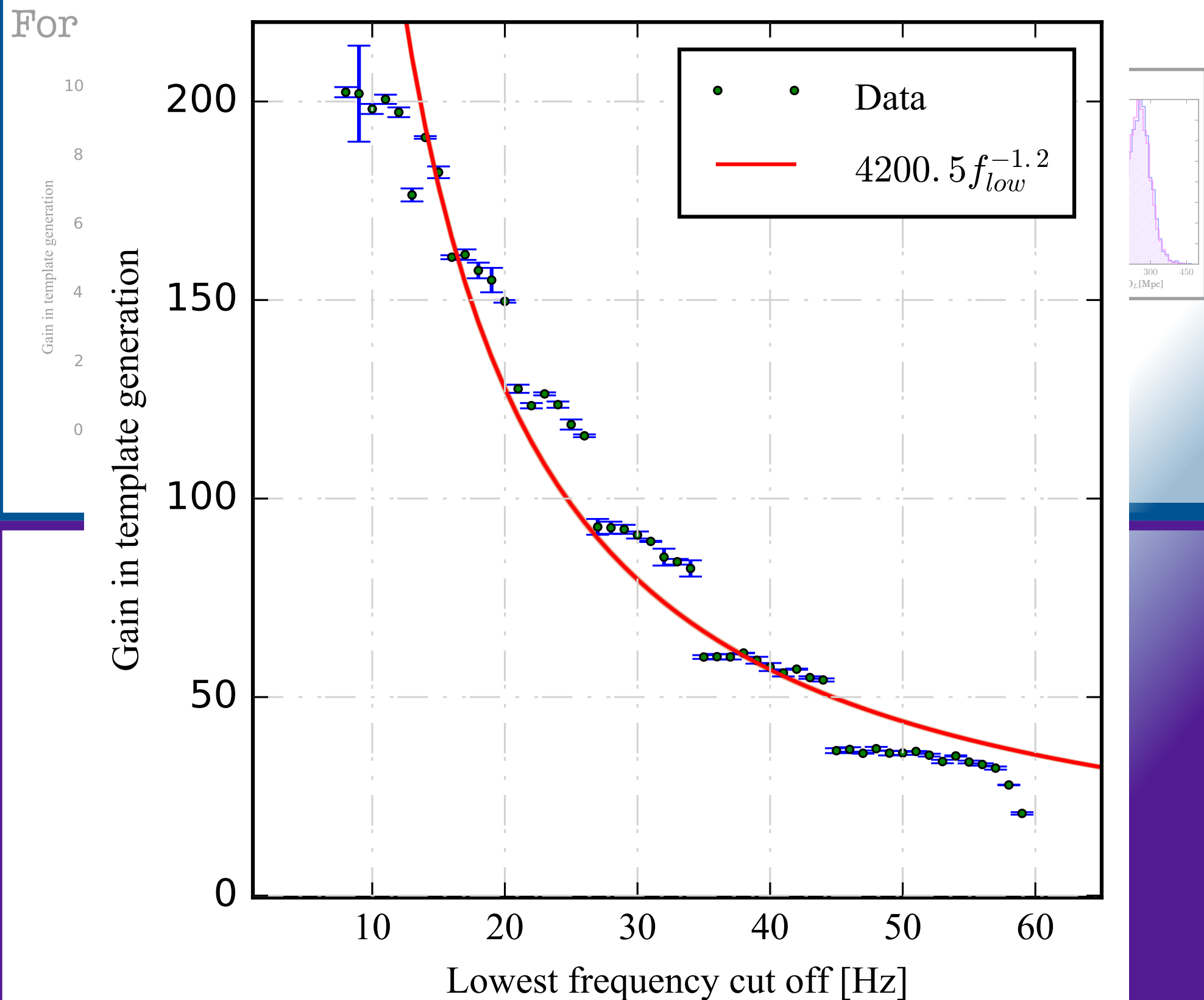
Why?

When?

Which?

What?

# RESULTS & MODEL DEPENDENCE





What?

Why?

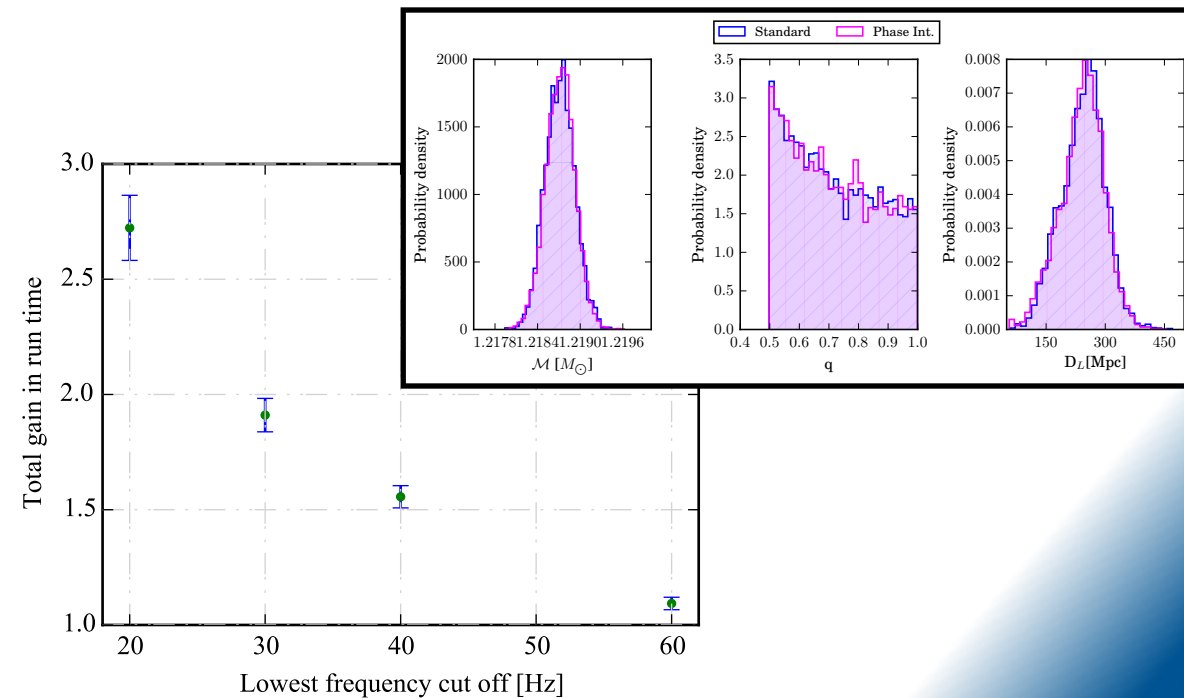
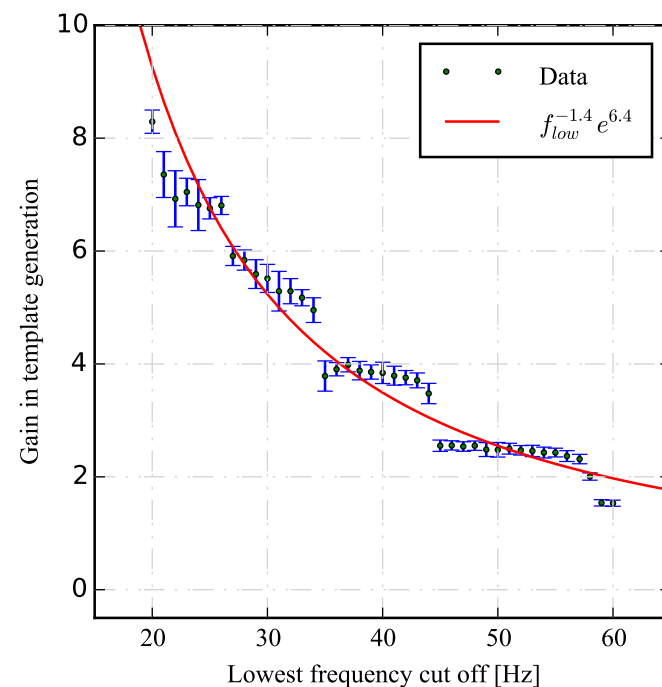
When?

Which?

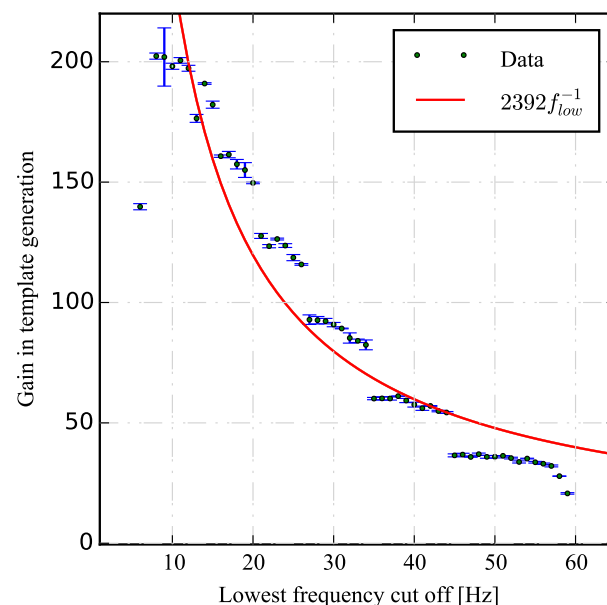
What?

# RESULTS & MODEL DEPENDENCE

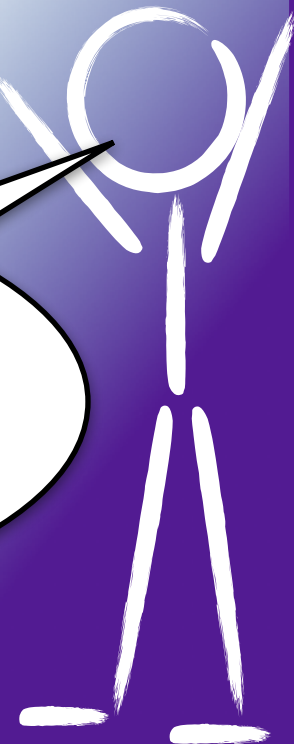
For one of the simplest waveform models : **TAYLORF2**



For a more sophisticated waveform model :  
**IMRPHENOMPv2**



With a OVERAL GAIN for  
Lowest frequency cut off  
of 40Hz of about a factor  
of **30**







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# SUMMARY

GW ASTRONOMY IS HERE ➡ NEED TO PROCESS DATA FASTER

NEW INSTRUMENTS ➡ BETTER LOW-FREQUENCY SENSITIVITY  
BETTER LOW-FREQUENCY SENSITIVITY ➡ MORE AND LONGER DATA

MULTI-BANDING + PHASE-INTERPOLATION  
➡ WIDE RANGE OF APPLICATIONS

MULTI-BANDING + PHASE-INTERPOLATION  
➡ GOOD RESULTS ALREADY

## WORK IN PROGRESS:

♦ can we improve the results? Probably yes  
on going further investigations

*Thank you for your attention*

GraWIToN



GW Initial Training Network

UNIVERSITY OF  
BIRMINGHAM



**EXTRA-SLIDES**

What?

Why?

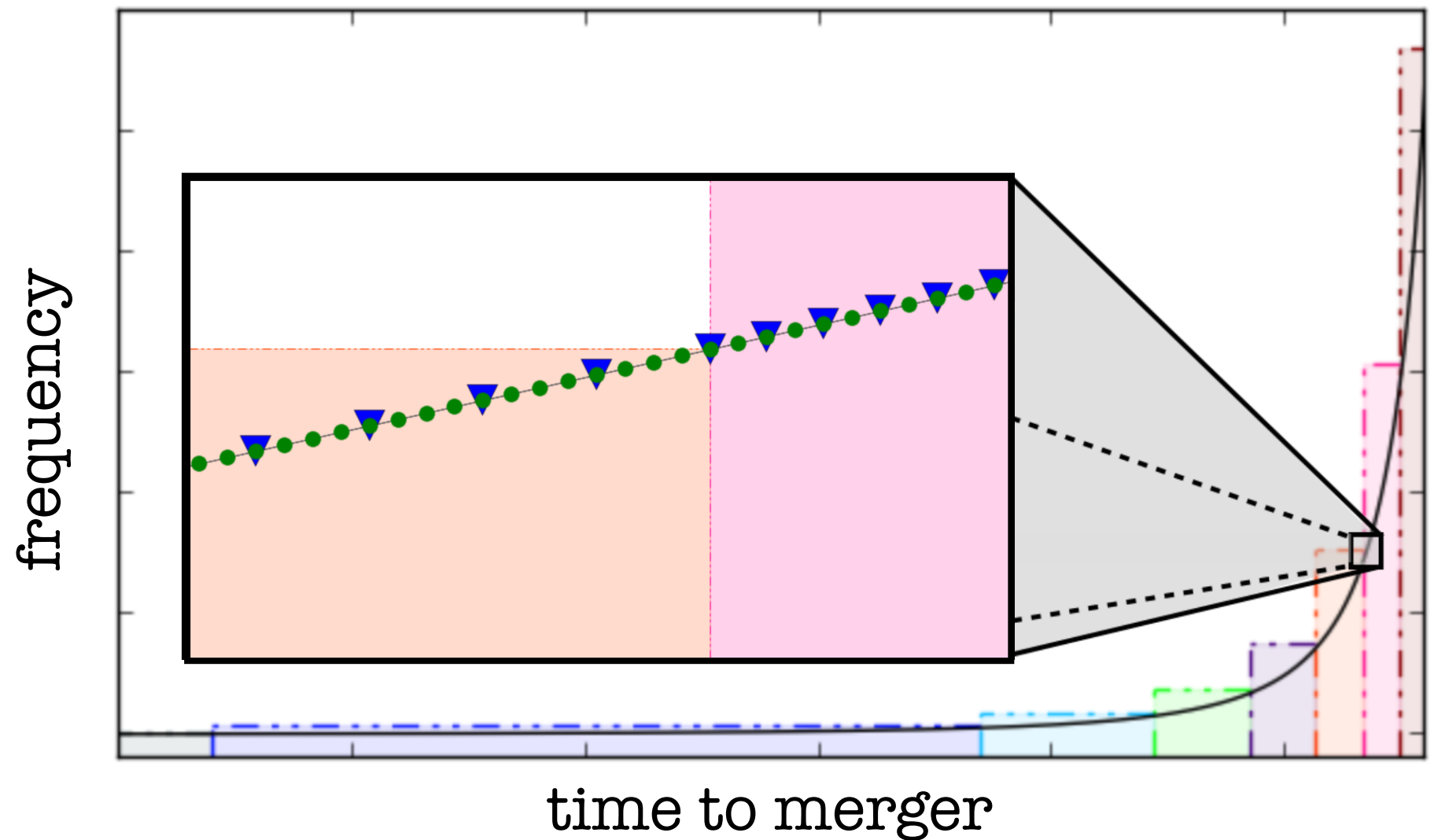
When?

Which?

What?

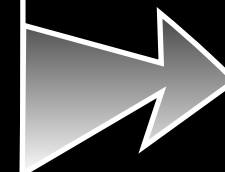
# FREQUENCY DOMAIN WAVEFORM

## EXAMPLE OF PRACTICAL RESULTS



In our case

DEFINITION OF  
FREQUENCY STEP  
IN BAND “l”



$$\delta f_l = 2^{n_l} \delta f_0$$



# DEFINITION OF THE FREQUENCY SET

$$\delta f(f) \leq \frac{1}{|t(f)| + t_{pad}}$$

$$\delta f_l = 2^{n_l} \delta f_0 \Rightarrow G = \frac{N_0}{N_T} = \frac{\sum_{l=0}^{NBands} N_{l,0}}{\sum_{l=0}^{NBands} N_{l,0} / 2^{n_l}} \quad \Bigg| \quad N_{l,0} = (f_{max,l} - f_{min,l}) \delta f_0^{-1}$$

**Step1** : Finding the maximum  $n_l$  allowed by the frequency range taken into account.

$$\delta f_{max} = \frac{1}{t_{pad}} \downarrow n_{max} = -\log_2 (\delta f_0 \cdot t_{pad})$$

**Step 2** : Finding the minimum necessary resolution.

$$\delta f_{min} = \frac{1}{|T_{tot}| + t_{pad}} \downarrow n_{min} = -\log_2 [\delta f_0 (|T_{tot}| + t_{pad})]$$

# DEFINITION OF THE FREQUENCY SET

**Step 3 :** Finding the frequency limits of each band.

It is convenient to start from the highest frequencies where  $n_l = n_{max}$ .

**Step 4 :** Define the frequencies at the extrema of all the new bins (now spaced by  $\delta f_l$ ) contained inside every frequency band. From  $f_{max}$  we define the new frequency set by  $f_i = f_{max} - i\delta f_{max} = f_{max} - i2^{n_{max}}\delta f_0$  until  $f_i \geq f_{lim_{max}}$ . When this last condition is no more satisfied a new band begins. Generalising the process, the frequencies inside the  $l$  band are fixed starting from the last frequency defined in the  $l - 1$  band and then keeping subtracting  $\delta f_l = 2^{n_l}\delta f_0$  until  $f_i \geq f_{lim_{\alpha}}$  is satisfied.



# STATIONARY PHASE APPROXIMATION

An alternative description can be performed in the frequency domain. [REF Cutler and Flanagan] In this context the waveform is usually computed by adopting and developing the *Stationary Phase Approximation* (SPA). This consists in the approximation to the Fourier transforms of the two GW polarisations. Given a function of time  $h(t) = \mathcal{A}(t) \cos \phi(t)$ , its Fourier transform :

$$\tilde{h}(f') = \int_{-\infty}^{\infty} h(t) e^{-i2\pi f' t} dt \quad (2.61)$$

can be estimated by the formula for positive frequencies  $f' \geq 0$ :

$$\tilde{h}(f') \approx \frac{1}{2} \mathcal{A}(t) \sqrt{\frac{dt}{df'}} e^{i(2\pi f' t(f') - \phi(f') - \pi/4)} \quad (2.62)$$

. where the time  $t$  is defined at  $d\phi/dt = 2\pi f'$ . This approximation holds whenever the variation of the angular velocity is much slower than itself, i.e.  $\frac{d^2\phi}{dt^2} \ll \left(\frac{d\phi}{dt}\right)$  and the time dependence of the function  $h(t)$  is led by the phase term, meaning that

$$\frac{dh}{dt}(t) = \frac{d\mathcal{A}}{dt} \cos \phi(t) - \mathcal{A}(t) \sin \phi(t) \frac{d\phi}{dt} \approx \mathcal{A}(t) \sin \phi(t) \frac{d\phi}{dt} \quad (2.63)$$

This last condition can be formulated by the requirement  $\frac{d \ln(\mathcal{A})}{dt} \ll \frac{d\phi}{dt}$ . At the Newtonian approximation, the amplitude  $\mathcal{A}$  is given by the expression (2.20), the frequency evolution is defined by equation (2.28) and the time as function of the frequency and the other physical parameters is given by (??). The time and phase as function of the be expressed as function of the gravitational frequency:

$$\begin{aligned} t(f) &= t_c - 5(8\pi f)^{-8/3} (GM)^{-5/3} c^5 \\ \phi(f) &= \phi_c - 2 [8\pi f GM]^{-5/3} c^5 \end{aligned} \quad (2.64)$$

These relations lead to the following estimation of  $\tilde{h}(f)$

$$\tilde{h}(f) = \frac{Q(\text{angles})}{d_L} \frac{(GM)^{5/6}}{c^{3/2}} f^{-7/6} e^{i\psi(f)} = A(f) e^{i\psi(f)} \quad (2.65)$$

where  $Q(\text{angles})$  is a function of the angle-parameters,  $A(f)$  is the amplitude and

$$\psi(f) = 2\pi f t_c - \phi_c + \frac{3}{4} c^5 (8\pi GM f)^{-5/3} - \frac{\pi}{4} \quad (2.66)$$

The frequency domain amplitude derived under the SPA can be intuitively understood by considering that the oscillatory terms contribute almost as  $\sim 1$  when in phase and almost 0 otherwise. How long is the time at which this condition is satisfied is determined by the term  $d^2\phi/dt^2 \approx df'/dt$ . A rough approximation can thus be given by requiring  $d^2\phi/dt^2 \Delta t^2 \sim 1$ , which can be used to determine the effective time duration in the Fourier transform definition:  $\Delta t \sim \sqrt{\frac{dt}{df'}}$ .

More accurate waveforms in frequency domain can be reached by developing the phase  $\psi(f)$  at higher orders of  $v$  or  $x$ .