Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

Sparsity of the Hawking Flux

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Outline

Black Holes Radiate Differently

- Black hole radiation usually compared to black body radiation

- However: Fundamental differences!

- Sparsity: 40-year-old result[1]

- Different view points:
  - $\lambda_{\text{thermal}} \gg r_H$

  - Long lifetime of black hole

  - $\boxed{\tau_{\text{loc}} \ll \tau_{\text{gap}} \; \longleftarrow \; \text{our focus}}$

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[1] E.g.: D. Page, PRD **13**(1976) 198; PRD **14**(1976) 3260; PRD **16**(1977) 2404

Measures of Sparsity

- Natural time-scales set by black body spectrum & emission probability

$$\mathrm{d}\Gamma = \frac{g}{(2\pi)^3} \; \frac{c\,(\hat{k}\cdot\hat{n})}{\exp\left((\epsilon(\vec{k}) - \mu)/k_{\mathsf{B}}T\right) + s} \; \mathrm{d}^3\vec{k}\,\mathrm{d}A, \quad \tau_{\mathsf{gap}} := \frac{1}{\Gamma}$$

- Many obvious choices, e.g.:

  -
  $$\eta_{\mathsf{peak\ number}} = \frac{\tau_{\mathsf{gap}}}{\tau_{\mathsf{peak\ number}}}$$

  -
  $$\eta_{\mathsf{peak\ energy}} = \frac{\tau_{\mathsf{gap}}}{\tau_{\mathsf{peak\ energy}}}$$

  -
  $$\eta_{\mathsf{average\ energy}} = \frac{\tau_{\mathsf{gap}}}{\tau_{\mathsf{average\ energy}}}$$

- Bolometric measure: $\eta_{\mathsf{binned}} = \frac{1}{\int_0^\infty \frac{2\pi}{ck}\frac{\mathrm{d}\Gamma}{\mathrm{d}k}\mathrm{d}k}$

What we look at

- Different types of black holes & particles in ray optics limit

- Black body approximation: Semi-analytic results

- Grey bodies: Numerical results

- Neglecting:
  - **Adiabaticity constraints:** Backreaction and how fast the space-time may change
    $\rightarrow$ *infrared cutoff*
  - **Phase space constraints:** Photon energy cannot exceed BH mass
    $\rightarrow$ *ultraviolet cutoff*
- Aiming for conservative estimates

Schwarzschild BH, Semi-analytic

For massless particles:

Type	$\eta_{\text{peak number}}$	$\eta_{\text{peak energy}}$	$\eta_{\text{avg. frequency}}$	η_{binned}
Bosons	$\frac{32\pi^2\left(2+W(-2e^{-2})\right)}{27g\zeta(3)}$	$\frac{32\pi^2\left(3+W(-3e^{-3})\right)}{27g\zeta(3)}$	$\frac{16\pi^6}{405g\zeta(3)^2}$	$\frac{128}{9g}$
$\approx$	$15.508/g$	$27.465/g$	$26.285/g$	$14.222/g$
Fermions	$\frac{128\pi^2\left(2+W(2e^{-2})\right)}{81g\zeta(3)}$	$\frac{128\pi^2\left(3+W(3e^{-3})\right)}{81g\zeta(3)}$	$\frac{224\pi^6}{3645g\zeta(3)^2}$	$\frac{256}{9g}$
$\approx$	$28.773/g$	$40.624/g$	$40.88/g$	$28.444/g$
Boltzmann	$\frac{64\pi^2}{27g}$	$\frac{32\pi^2}{9g}$	$\frac{32\pi^2}{9g}$	$\frac{64\pi^2}{27g}$
$\approx$	$23.395/g$	$35.092/g$	$35.092/g$	$23.395/g$

Things can be summarised, e.g.:

$$\eta_{\text{peak number}} = \frac{\lambda_{\text{thermal}}^2}{A_{\text{H}}} \frac{2 + W(2s\ e^{-\mu-2})}{2\text{Li}_3(-s\ e^{-\mu})/(-s)}, \quad s \in \{-1,0,1\}$$

Generalising Semi-Analytics

- For Reissner-Nordström & Kerr in absence of super-radiance:

$$\eta = \eta_{\text{Schwarzschild}} \times \frac{r_+^2}{(r_+ - r_-)^2} \geq \eta_{\text{Schwarzschild}}$$

- Dirty black holes:

$$\eta = \eta_{\text{Schwarzschild}} \times \frac{e^{\phi_H}}{1 - 8\pi G_N \rho_H r_H^2 / c^4}, \quad \phi_H = \frac{4\pi G_N}{c^4} \int_{r_H}^\infty \frac{(\rho - p_r)\, r}{1 - 2m(r)/r}\, \mathrm{d}r \geq 0$$

- Particle rest mass:

$$\frac{\int_0^\infty \frac{x^2}{\exp(\sqrt{z^2 + x^2}) \mp 1}\, \mathrm{d}x}{\int_0^\infty \frac{x^2}{\exp(x) \mp 1}\, \mathrm{d}x} \leq 1 \quad \implies \quad \eta \geq \eta_{\text{Schwarzschild}}$$
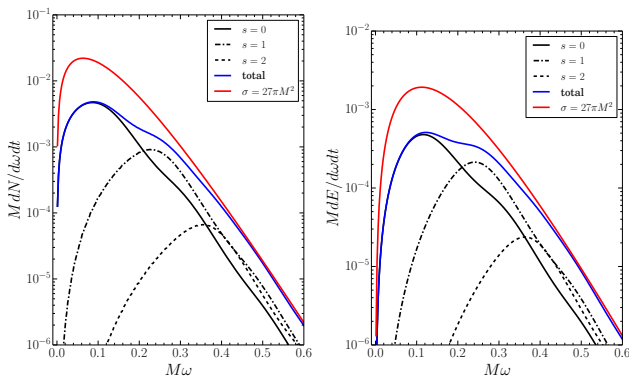
- Extending to graviton results for Tangherlini metric[2]:

$$\eta_{\text{Tang,grav}} = \frac{1}{(d-3)!\, 2\pi} \frac{\Gamma(\frac{d-2}{2})}{\pi^{(d-4)/2}} \frac{(d-2 + W((d-2)se^{-\mu - d + 2}))}{\frac{\mathrm{Li}_{d-1}(-se^{-\mu})}{(-s)}} \frac{\lambda_{\text{thermal}}^{d-2}}{g A_H} \overset{d \lesssim 10}{>} 1$$

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[2] Unpublished, compare to S. Hod, Phys. Lett. B, **756**, 133-136 (2016)

Numerical Results

Ray optics limit $\longrightarrow$ Turn Regge-Wheeler equation into Shabat-Zakharov system, use product calculus[3]

Grey body factors make things even sparser!

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[3]F. Gray, M. Visser, arXiv:1512.05018, consistent with Teukolsky & Page

Super-Radiance

- q and $L \to$ chemical potentials/ergo-regions

- For $\omega < \omega_{\text{crit}}$: $G_{\ell,m}$ and $\langle N \rangle_\epsilon$ now $< 1 \Rightarrow$ Super-radiance

- Both effects act in parallel

- Therefore (here for angular momentum and binned η):

$$\frac{1}{\eta_{\text{super-radiance}}} + \frac{1}{\eta_{\text{Hawking}}} = 2\pi \sum_{\ell,m} \int_0^{\omega_{\text{crit}}} G_{\ell,m} N_\epsilon \frac{d\omega}{\omega} + 2\pi \sum_{\ell,m} \int_{\omega_{\text{crit}}}^\infty G_{\ell,m} N_\epsilon \frac{d\omega}{\omega}$$

- For $\kappa \to 0$, super-radiance dominates:

$$\eta_{\text{super-radiant}} = \mathcal{O}\left(\epsilon^0\right); \qquad \eta_{\text{Hawking}} = \mathcal{O}\left(\epsilon^{2s+2}\right) \ggg 1$$

since $\epsilon := \hbar\omega_{\text{crit}}/(k_B T_H) \ggg 1$

Conclusion and Outlook

Summary

> Even conservative semi-analytical
> estimates are lower bounds for η_{Hawking}!

- $1 \ll \eta_{\text{semi-analytic, Schwarzschild}} \leq \eta_{\text{semi-analytic, general}}$
- After some care with g: $\eta_{\text{Boson}} < \eta_{\text{Boltzmann}} < \eta_{\text{Fermion}}$
- $\implies$ **Sparse** flux!
- Unlike black/grey bodies, Hawking radiation is a long 2-body decay chain

Outlook

- Analogue black holes?
- Correlations in interstitial gaps?

Thank you!

Questions?

Schwarzschild – Numerical Outline

- Wave equation on curved background $\rightarrow$ stationary, 1D-Schrödinger equation in r^* with Regge-Wheeler potential

$$V(r^*) = \left[1 - \frac{2M}{r(r^*)}\right] \left[\frac{\ell(\ell+1)}{r(r^*)^2} + \frac{(1-s^2)2M}{r(r^*)^3}\right]$$

- Rephrase Schrödinger eqn. as Shabat-Zakharov system
- Use (numerical) product calculus to solve for Bogoliubov coefficients (F. Gray, M. Visser, arXiv:1512.05018):

$$\begin{pmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{pmatrix} = \prod_{-\infty}^{\infty} \left(\mathcal{I} - \frac{i}{4x}V(u^*) \begin{pmatrix} 1 & \exp(-4ixu^*) \\ -\exp(4ixu^*) & -1 \end{pmatrix}\right) \mathrm{d}u^*$$

with $u = r/2M, u^* = r^*/2M$

- Check consistency with earlier results by Teukolsky & Page: $\checkmark$

Shabat-Zakharov System

- Start with a wave equation on curved background:

$$\nabla_\mu \nabla^\mu \psi = 0$$

- Rephrase resulting Regge-Wheeler equation with the ansatz

$$\psi(x) = a(x)\frac{\exp(+i\varphi)}{\sqrt{\varphi'}} + b(x)\frac{\exp(-i\varphi)}{\sqrt{\varphi'}}$$

and gauge condition

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{a}{\sqrt{\varphi'}}\right)e^{+i\varphi} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{b}{\sqrt{\varphi'}}\right)e^{-i\varphi} = 0$$

- With $\omega(x)^2 := E - V(x)$ and $\rho := \varphi'' + i(\omega(x)^2 - (\varphi')^2)$ get S-Z equations:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} a(x) \\ b(x) \end{pmatrix} = \frac{1}{2\varphi'}\begin{pmatrix} i\,\mathrm{Im}(\rho) & \rho\exp(-2i\varphi) \\ \rho^*\exp(2i\varphi) & -i\,\mathrm{Im}(\rho) \end{pmatrix}\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$$

Product Calculus

- Formally solve S-Z equations by path-ordered integration
$$\begin{pmatrix} \alpha & \beta^* \\ \beta & \alpha^* \end{pmatrix} = \mathcal{P} \exp \left(-\frac{i}{4x} \int_{-\infty}^{\infty} V(u^*) \begin{pmatrix} 1 & \exp(-4ixu^*) \\ -\exp(4ixu^*) & -1 \end{pmatrix} \mathrm{d}u^* \right)$$

- Equivalent to product calculus:

$$\int_a^b f(x)\mathrm{d}x = \lim_{N\to\infty} \sum_{k=0}^{N-1} f(t_k)\Delta x_k$$

$$\leftrightarrow \quad \prod_a^b \left(\mathcal{I} + A(x)\ dx \right) \equiv \lim_{N\to\infty} \prod_{k=0}^{N-1} \left(\mathcal{I} + A(x_k^*)\Delta x_k \right)$$

- Many approaches to numerical evaluation, including:
  - Peano-Baker series
  - Simpson rule
  - Truncating the defining limits
  - . . .