Spacetime Emergence
Through A Geometric Renormalization Method

Saeed Rastgoo (UAM-I, Mexico)

In collaboration with
Manfred Requardt (University of Göttingen)

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The Method in a Nutshell

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4. Lower chain: coarse graining chain discrete spaces. Upper chain: the continuum limit chain.

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- end in a fixed point, a set of accumulation points or converge under $\mathcal{K}$ (non-generic senario).

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Given metric spaces $X, Y \in \mathcal{S}$, a map $f : X \to Y$, where $\exists \lambda \geq 1, \epsilon \geq 0$ such that $\forall x_1, x_2 \in X$

$$\frac{1}{\lambda} d_X(x_1, x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \epsilon, \qquad \forall x_1, x_2 \in X$$

and

$$\forall y \in Y : \; \exists x \in X : \; d_Y(y, f(x)) \leq C.$$

is a quasi-isometry, i.e.

- distance of the images under f, within a factor λ, and up to a constant, of their original distances, and
- every point $y \in Y$ lies within a constant distance $C \geq 0$ of an image point.

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It is a rough isometry when $\lambda = 1$.

Examples of $\mathcal{K}$

k-local insertion/deletion of edges: the k-neighborhood of vertices of a graph G such that the k-neighborhood remains invariant. Pure quasi-isometry.

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$G \longrightarrow \mathcal{C}(G)$ "clique graph":

- Vertices $V(\mathcal{C}(G))$: maximal subsimplexes (cliques) of G,
- An edge between two of vertices of $\mathcal{C}(G)$ if cliques have non-zero vertex overlap (in G).

Rough isometry.

A Measure of Similarity (and Convergence): d_{GH}

Gromov-Hausdorff distance $d_{GH}(X, Y)$ between two compact metric spaces (can be extended to $\mathcal{S}$):

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- $\mathcal{K}$ is rough isometry, then $d_{GH}(X, Y) =$ finite. Spaces are, to some extent, structurally similar.

Rescaling ϕ_λ and Continuum Limit Properties

The rescaling map, ϕ_λ, on a metric space (X, d_X)

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λ parametrizes the distance between the points on the different length scales.

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- $\lim_{\lambda\to\infty} \phi_\lambda$ reveals the fine structure of X by magnifying the infinitesimal neighborhoods of the points of X,
- $\lim_{\lambda\to 0} \phi_\lambda$ corresponds to the large scale structure of X:

$$\lim_{\lambda\to\infty} \phi_\lambda\left((X, d_X)\right) = \lim_{\lambda\to\infty} (X, \lambda d_X) = \left(X_\infty, d_{X,\infty}\right),$$

important for us.

Some Properties of the Continuum Limit

If spaces are purely quasi-isometric, i.e. $d_{GH}(X, Y) = \infty$, then

$$d_{GH}(X_\infty, Y_\infty) = \infty,$$

i.e. they also have different continuum limits.

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Set $\left\{(X', d_{X'}) \middle| d_{GH}(X', X) < \infty\right\}$ is the basin of attraction, for the attractor $(X_\infty, d_{X,\infty})$, under the evolution map ϕ_λ. They all have the same unique continuum limit $(X_\infty, d_{X,\infty})$.

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$(X_\infty, d_{X,\infty})$ is scale invariant under ϕ_λ i.e. $d_{GH}(X_\infty, \lambda X_\infty)=0$.

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with $G_i = \mathcal{K}^i(G_0)$, etc.

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- However, they are homeomorphic; can even be chosen to be the same topological space.

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- However, they are homeomorphic; can even be chosen to be the same topological space.
- Implies that different levels of spacetime will have different metric even if they are the same set.

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Coarse graining goes on until either:

1. $\mathcal{K}$ turns into (phase transition to) a rough isometry (due to the change of structure of graphs),

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with $G_i = \mathcal{K}^i(G_0)$, etc.

Coarse graining goes on until either:

1. $\mathcal{K}$ turns into (phase transition to) a rough isometry (due to the change of structure of graphs),
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The Full Picture

Combination of two operations

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    1. if the spaces are uniformly compact, the Gromov's compactness theorem shows they convergence with respect to d_{GH}. Not generic.

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For graphs with polynomial growth, dimension

$$D(G) = \lim_{r \to \infty} \frac{\log \beta(G, v_i, r)}{\log r}$$

coincides with usual dim for embedded spaces and lattices.

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- Graphs with locally finite vertex degree, being connected, and vertex transitive: integer dimension.
- D stable under quasi-isometry $\mathcal{K}$.
- If we want the change of D under renormalization: not quasi-isometric $\mathcal{K}$ but translocal.

Summary

- A geometric RG: using notions of isometric coarse graining, and rescaling.
- Coarse graining goes on until a phase transition occurs, quasi $\longrightarrow$ rough isometry.
- Quasi-isometric spaces are structurally different, have different continuum limits.
- Rough isometric spaces are structurally similar, have the same unique continuum limit.
- Dimension stable under k-local RG.
- Change of dimension under RG with translocal operations.

Rescaling ϕ_λ and continuum limit properties

Take lattice $\mathbb{Z}^n$ embedded in $\mathbb{R}^n$, take the scaling limit

$$\phi_l : (\mathbb{Z}^n, d_{\mathbb{Z}^n}) \longmapsto (\mathbb{Z}^n, \lambda d_{\mathbb{Z}^n}), \qquad \lambda = 2^{-l}$$

with $d_{\mathbb{Z}^n}$ a suitable metric on $\mathbb{Z}^n$. Then

$$\lim_{\lambda \to 0} (\mathbb{Z}^n, \lambda d_{\mathbb{Z}^n}) = \mathbb{R}^n,$$

in pointed GH-sense. For a fixed ball around $x = 0$, and for $l \to \infty$ the ball is more and more filled with points stemming from lattices having edge length 2^{-l}.