Geodesic completeness and lack of strong singularities in loop quantization of Kantowski-Sachs spacetime

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Introduction and Motivation

- It has been shown that the big bang singularity is replaced by a big bounce in loop quantization of isotropic and homogeneous models. (Ashtakar, Pawlowski, Singh 2006)

- For all other types of singularities in isotropic and homogeneous models, and in effective dynamics of LQC (Singh, 2009):
  - Energy density, expansion and shear scalars are generically bounded.
  - Curvature invariants are bounded except certain pressure singularities (called sudden singularities).
  - Strong singularities absent, weak singularities possible.

- These results have been extended to Bianchi-I (homogeneous and anisotropic) model. (P. Singh, 2012)

- We extend it further to Kantowski-Sachs model (homogeneous and anisotropic spacetime) in effective dynamics and show that:
  - Expansion and shear scalars are generically bounded.
  - Energy density is bounded for all finite proper time.
  - Curvature invariants are bounded, for all finite proper time, except certain pressure singularities.
  - All finite time strong singularities absent, weak singularities possible.

Classical Dynamics of Kantowski-Sachs spacetime

- The Kantowski-Sachs Metric : homogeneous and anisotropic

$$ds^2 = -N(t)^2 dt^2 + g_{xx}(t) dx^2 + g_{\Omega\Omega}(t) \left(d\theta^2 + \sin^2\theta d\phi^2 \right).$$

is singular in classical GR with the Schwarzschild interior as the following special case

$$N(t)^2 = \left(\frac{2m}{t} - 1 \right)^{-1}, \quad g_{xx}(t) = \left(\frac{2m}{t} - 1 \right), \quad g_{\Omega\Omega}(t) = t^2$$

- Expressing in terms of symmetry-reduced Ashtekars triad variables

$$g_{xx} = \frac{p_b^2}{L_o^2 p_c}, \quad g_{\Omega\Omega} = p_c$$

the conjugate connection variables are denoted by b and c.
- The non-vanishing Poisson brackets between these variables are given by

$$\{b, p_b\} = G\gamma, \ \ \{c, p_c\} = 2G\gamma$$

- Singularity occurs when:

$$p_b = 0 \quad \text{and/or} \quad p_c = 0$$

Loop Quantization of Kantowski-Sachs Model

- Loop Quantum Gravity(LQG) is canonical quantization of GR using Ashtekar variables.
- Loop Quantum Cosmology(LQC)
  - application of LQG in cosmology.
  - current approach : directly quantize symmetry reduced models of physical relevance, i.e. symmetry reduction is done before quantizing.
- Effective Hamiltonian obtained from improved-dynamics quantization (Boehmer and Vandersloot, 2007)

$$\mathcal{H} = -\frac{p_b\sqrt{p_c}}{2G\gamma^2\Delta}\left[2\sin(b\delta_b)\sin(c\delta_c) + \sin^2(b\delta_b) + \frac{\gamma^2\Delta}{p_c}\right] + 4\pi p_b\sqrt{p_c}\rho$$

where,

$$\delta_b = \sqrt{\frac{\Delta}{p_c}}, \quad \delta_c = \frac{\sqrt{\Delta p_c}}{p_b}$$

and the constants $\gamma = 0.2375$ and $\Delta = 4\sqrt{3}\pi\gamma l_p^2$

Bounds on triad variables in LQC

- Dynamical equations can be obtained from the effective Hamiltonian using Hamilton's equations, e.g.

$$\dot{p}_c = -2G\gamma \frac{\partial \mathcal{H}}{\partial c} = \frac{2p_c}{\gamma\sqrt{\Delta}} \sin(b\delta_b) \cos(c\delta_c)$$

  similarly for p_b, b and c.

- These dynamical equations reduce to classical GR equations in the classical limit $\Delta \to 0$.
- Dynamical equations for p_b and p_c yield that

$$\frac{\dot{p}_b}{p_b} \text{ and } \frac{\dot{p}_c}{p_c} \quad \text{are generically bounded.}$$

- From the equations for $\dot{p}_b$ and $\dot{p}_c$, it can be shown analytically that

$$0 < p_c(t) < \infty \quad \text{and} \quad 0 < p_b(t) < \infty$$

  for finite time evolution.

Energy-density, Expansion-scalar and Shear-scalar

- The expansion and shear scalar are bounded (A. Joe and P. Singh, 2015).

$$\theta = \frac{\dot{p}_b}{p_b} + \frac{\dot{p}_c}{2p_c}. \quad \sigma^2 = \frac{1}{3}\left(\frac{\dot{p}_c}{p_c} - \frac{\dot{p}_b}{p_b}\right)^2$$

- The energy density is obtained from vanishing of the Hamiltonian constraint

$$\rho = \frac{1}{8\pi G\gamma^2\Delta}\left[2\sin(b\delta_b)\sin(c\delta_c) + \sin^2(b\delta_b) + \frac{\gamma^2\Delta}{p_c}\right]$$

  In earlier works, it was numerically demonstrated in specific cases that energy density is dynamically bounded. With our new result for analytical bounds on p_b and p_c, energy density is in general bounded for all finite times.

- Volume $V = 4\pi p_b\sqrt{p_c}$ also remains non-zero and finite for all finite times.

$$0 < V < \infty$$

Curvature Invariants

- Do bounds on expansion scalar (θ), shear scalar (σ^2), and energy density (ρ) imply that there are no singularities?

- Not necessarily. For example, the expressions for Ricci scalar and the square of the Weyl scalar are,

$$
\begin{aligned}
R &= 2\frac{\ddot{p}_b}{p_b} + \frac{\ddot{p}_c}{p_c} + \frac{2}{p_c} \\
C_{abcd}C^{abcd} &= \frac{1}{3}\left[3\frac{\dot{p}_c}{p_c}\left(\frac{\dot{p}_b}{p_b} - \frac{\dot{p}_c}{p_c}\right) - 2\left(\frac{\ddot{p}_b}{p_b} - \frac{\ddot{p}_c}{p_c}\right) - \frac{2}{p_c}\right]^2
\end{aligned}
$$

- The curvature invariants depend on 2nd derivatives $\frac{\ddot{p}_b}{p_b}$, $\frac{\ddot{p}_c}{p_c}$, which in turn depend on derivatives of energy density $\frac{\partial \rho}{\partial p_b}$ and/or $\frac{\partial \rho}{\partial p_c}$.

Curvature Invariants (continued)

- Looking at the second derivatives, for example:

$$
\begin{aligned}
\frac{\ddot{p}_b}{p_b} = & \frac{\cos(b\delta_b)\cos(c\delta_c)}{p_c} + \frac{\cos^2(b\delta_b)}{\gamma^2\Delta}(\sin(b\delta_b)+\sin(c\delta_c))^2 \\
& -\frac{4\pi}{\gamma^2\sqrt{\Delta}}\frac{(cp_c - bp_b)}{V}\cos(c\delta_c)\left(\sin(c\delta_c)+\sin^3(b\delta_b)\right) \\
& +4\pi G\left(2p_c\frac{\partial\rho}{\partial p_c}\cos(b\delta_b)\cos(c\delta_c) - p_b\frac{\partial\rho}{\partial p_b}\sin(b\delta_b)\sin(c\delta_c)\right. \\
& \left. +p_b\frac{\partial\rho}{\partial p_b}\cos(2b\delta_b)\right)
\end{aligned}
$$

- The following quantity remains unchanged from classical to effective dynamics:

$$\frac{d}{dt}(cp_c - bp_b) = \frac{\gamma p_b}{\sqrt{p_c}} + G\gamma V\left(2p_c\frac{\partial\rho}{\partial p_c} - p_b\frac{\partial\rho}{\partial p_b}\right)$$

  This was found to be zero in case of Bianchi-I model.
- All possible divergences come from the derivatives of energy density $\frac{\partial\rho}{\partial p_b}$, $\frac{\partial\rho}{\partial p_c}$.

Curvature Invariants (continued)

- Conclusion: Curvature invariants may diverge under certain specific circumstances: if the derivatives $\frac{\partial \rho}{\partial p_b}$ and/or $\frac{\partial \rho}{\partial p_c}$ diverge at a finite value of ρ, θ and σ^2 at a non-zero volume.

- These derivatives are related to the pressure. For example, in case of matter with vanishing anisotropic stress,

$$p_b \frac{\partial \rho}{\partial p_b} = 2 p_c \frac{\partial \rho}{\partial p_c} = -\rho - P \qquad (1)$$

  so divergences in $\frac{\partial \rho}{\partial p_b}$ and/or $\frac{\partial \rho}{\partial p_c}$ occur due to pressure divergences.

- The potential divergences in curvature invariants hints that there may be some potential singularities.

Geodesic Completeness

- However, geodesics never break down : Kantowski-Sachs spacetime is geodesically-complete in the effective dynamics of LQC.

$$
\begin{aligned}
x' &= C_x \frac{L_o^2 p_c}{p_b^2} \\
\phi' &= \frac{C_\phi}{p_c} \\
t'^2 &= \epsilon + C_x^2 \frac{L_o^2 p_c}{p_b^2} + \frac{C_\phi^2}{p_c}
\end{aligned}
$$

  where C_x and C_ϕ are constants of integration, and ϵ is 1 for timelike geodesics and 0 for null geodesics.

  Due to bounds on p_b and p_c, geodesics are defined everywhere.

- Geodesic completeness indicates that any potential singularities may turn out to be weak singularities. We follow up with the analysis of strength of singularities for further insights.

Strength of Singularities Analysis

- It has been conjectured that any physical singularity must be a strong curvature singularity. (Tipler, Clarke and Ellis, 1980).
- A strong curvature singularity : any in-falling objects are crushed to zero volume irrespective of the properties of the object. (Ellis and Schmidt,1977).

  formulated in precise mathematical form by Tipler (1977), and Krolak (1980s)
- The necessary condition (Clarke and Krolak, (1985)):
  If a singularity is a strong curvature singularity, then, for a timelike (or null) geodesic running into the singularity, the integral

$$K^i_j = \int_0^\tau d\tau' |R^i_{4j4}(\tau')|$$

does not converge as $\tau \to \tau_o$, where τ_o is the position of singularity.

- All the non-zero Riemann tensor components have terms that can be classified into two types:
  - Terms of the type $f(p_b, p_c)(\frac{\dot{p}_c}{p_c})^m(\frac{\dot{p}_b}{p_b})^n$, m, n positive integers

    Since $\frac{\dot{p}_b}{p_b}$ and $\frac{\dot{p}_c}{p_c}$ are bounded functions,

    p_b and p_c shown to be finite for finite time evolution, then

$$\int_0^{\tau_o} d\tau f(p_b, p_c)\left(\frac{\dot{p}_c}{p_c}\right)^m\left(\frac{\dot{p}_b}{p_b}\right)^n$$

    is finite for finite final time τ_o.

  - Terms of type $g(p_b, p_c)\frac{\ddot{p}_c}{p_c}$ or $g(p_b, p_c)\frac{\ddot{p}_b}{p_b}$

$$\begin{aligned} \int g(p_b, p_c)\frac{\ddot{p}_c}{p_c}d\tau &= g(p_b, p_c)\frac{\dot{p}_c}{p_c} - \int \dot{p}_c\left(\frac{d}{d\tau}\frac{g(p_b, p_c)}{p_c}\right)d\tau \\ &= f_1(p_b, p_c, \frac{\dot{p}_c}{p_c}, \frac{\dot{p}_b}{p_b}) - \int f_2(p_b, p_c, \frac{\dot{p}_c}{p_c}, \frac{\dot{p}_b}{p_b})d\tau \end{aligned}$$

    hence finite if the integral is over a finite time.

- It turns out that this integral is bounded in case of Kantowski-Sachs in LQC for finite proper times. The potential singularities due to divergences in curvature invariants turn out to be weak.

Summary

We find that Kantowski-Sachs spacetime in Bohmer-Vandersloot quantization (LQC) has following properties:

- Expansion and shear scalar are generically bounded for all time.
- Energy density is bounded for all finite times.
- Volume stays non-zero and finite for all finite times.
- Curvature Invariants are bounded for all finite time except for weak pressure singularities.
- All finite time strong curvature singularities are removed.
- Reduces to classical spacetime in classical limit.