The global nonlinear stability of Minkowski space for the f(R) theory of modified gravity

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- Einstein equations for self-gravitating matter
- Cauchy developments from initial data sets on a spacelike hypersurface
- Global dynamics of the matter content
- Global geometry of the spacetime

Joint work with Y. Ma (Xian, China)

OBJECTIVE

Nonlinear wave-Klein-Gordon systems

- ‣ Einstein-massive scalar field system
- ‣ The theory of modified gravity (additional constraint equations)
- ‣ Scalar-tensor theories

The nonlinear stability of Minkowski spacetime

- ‣ The Klein-Gordon potential drastically modifies the global dynamics.
- ‣ Exclude dynamically unstable, self-gravitating massive modes

  (trapped surfaces, black hole)
- ‣ Field equations and a prescribed initial data set
  - ‣ a small perturbation
  - ‣ a spacelike hypersurface in Minkowski space
  - ‣ sufficiently small massive field

Global nonlinear stability problem

- ‣ Perturbations disperse in timelike directions and the spacetime is timelike geodesically complete.
- ‣ Time decay estimates for systems of coupled wave-Klein-Gordon equations
- ‣ Hyperboloidal Foliation Method   (PLF-YM, Monograph, 2014)

References

- P.G. LeFloch and Y. Ma
    - *The hyperboloidal foliation method,* World Scientific Press, 2014

      ArXiv:1411.4910

    - *The nonlinear stability of Minkowski space for self-gravitating massive fields*
        - *A wave-Klein-Gordon model*   ArXiv:1507.01143
        - *Analysis of the Einstein equations*   ArXiv:1511.03324
        - *Analysis of the $f(R)$-theory of modified gravity*   ArXiv:1412.8151

- Related (earlier as well as more recent) works
    - Hyperboloidal foliations of Minkowski spacetime
        - Klainerman (1985) for quasilinear Klein-Gordon equations, revisited by Hormander (1997)
        - Donninger and Zenginoglu (2014): a nondispersive decay property for the cubic wave equation.

    - Hyperboloidal foliations of curved spacetimes
        - Friedrich (1983): hyperboloidal foliations associated with the vacuum Einstein equations (conformal vacuum field equations), investigated numerically by Zenginoglu (2008) and Moncrief and Rinne (2009).
        - Q. Wang, Einstein equations with massive scalar fields, Nov. 2015.
        - D. Fajman, J. Joudioux, and J. Smulevici, relativistic transport equations, Nov. 2015

SELF-GRAVITATING MASSIVE FIELDS

Massive scalar field with potential $U(\phi)$ (minimally coupled)

$$T_{\alpha\beta} := \nabla_\alpha\phi\nabla_\beta\phi - \left(\frac{1}{2}\nabla_\gamma\phi\nabla^\gamma\phi + U(\phi)\right)g_{\alpha\beta}$$

Einstein-**Klein**-**Gordon system** for the unknown $(M, g_{\alpha\beta}, \phi)$

$$\Box_g\phi - U'(\phi) = 0$$

$$R_{\alpha\beta} - 8\pi\left(\nabla_\alpha\phi\nabla_\beta\phi + U(\phi)\,g_{\alpha\beta}\right) = 0$$

for instance $U(\phi) = \frac{c^2}{2}\phi^2$

FIELD EQUATIONS OF MODIFIED GRAVITY

- Long history in physics: Weyl 1918, Pauli 1919, Eddington 1924, etc.

- Alternative theories of gravity: the gravitational field is mediated by one or more scalar fields.

- A function $f(R) \simeq R$ of the scalar curvature
  $$f(R) = R + \kappa R^2, \qquad f(R) = R + \kappa R^n, \qquad f(R) = R + \frac{\kappa R^n}{R_\star^n + R^n}$$

- Motivations from cosmology; broad literature in physics
    - accelerated expansion of the Universe; structure formation
    - without adding unknown forms of dark matter, dark energy, etc.
    - formation of structures in the Universe (galaxies, etc.)

Action

$$\mathcal{A}_{\mathsf{MG}}[\phi, g] := \int_M \left(f(R_g) + 16\pi L[\phi, g] \right) dV_g$$

- Prescribed function $f : \mathbb{R} \to \mathbb{R}$
  - $f(R) = R + \kappa \left(\frac{R^2}{2} + \mathcal{O}(\kappa R^3) \right)$
  - sign of $\kappa := f''(0) > 0$ essential for global stability

Curvature tensor of modified gravity

$$N_{\alpha\beta} := f'(R_g)\, R_{\alpha\beta} - \frac{1}{2} f(R_g) g_{\alpha\beta} + \left(g_{\alpha\beta} \square_g - \nabla_\alpha \nabla_\beta \right) \left(f'(R_g) \right)$$

$$= f'(R_g)\, G_{\alpha\beta} - \frac{1}{2} \left(f(R_g) - R_g f'(R_g) \right) g_{\alpha\beta} + \left(g_{\alpha\beta} \square_g - \nabla_\alpha \nabla_\beta \right) \left(f'(R_g) \right)$$

$$N_{\alpha\beta} = 8\pi T_{\alpha\beta}[\phi, g]$$

- *Fourth-order* derivatives of the unknown metric

- When f is linear, $N_{\alpha\beta}$ reduces to $G_{\alpha\beta}$.

THE REFORMULATION OF THE FIELD EQUATION

$$N_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

- Fourth-order system (with no specific PDE type)
  while Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ are second-order

- *Conformal transformation* leading to a third-order system

- *Wave coordinates* $\Box_g x^{\alpha} = 0$ associated with the spacetime metric g
  while Einstein equations are
  hyperbolic with differential constraints

- *Augmented formulation*
  - spacetime scalar curvature taken as an *independent variable*
  - leading to a second-order system of *nonlinear wave-Klein-Gordon equations*

- Analogy with the structure of the Einstein-massive field system

THE CONFORMAL TRANSFORMATION

Gravitational curvature tensor of modified gravity

$$N_{\alpha\beta} = f'(R_g)\,R_{\alpha\beta} + \Big(g_{\alpha\beta}\,\Box_g - \nabla_\alpha\nabla_\beta\Big)f'(R_g) - \frac{1}{2}f(R_g)g_{\alpha\beta} = 8\pi\,T_{\alpha\beta}$$

Hessian of the scalar curvature $\nabla_\alpha\nabla_\beta f'(R_g)$ (fourth-order term)

Conformally equivalent metric

$$g^\dagger{}_{\alpha\beta} := e^{2\rho}\,g_{\alpha\beta}, \qquad \rho := \tfrac{1}{2}\ln f'(R_g)$$

Conformal transformation for the Ricci curvature

$$R_{\alpha\beta} = R^\dagger{}_{\alpha\beta} + \big(2\,\nabla_\alpha\nabla_\beta\rho + g_{\alpha\beta}\,\Box_g\rho\big) + 2\big(-\nabla_\alpha\rho\nabla_\beta\rho + g_{\alpha\beta}\,\nabla^\gamma\rho\nabla_\gamma\rho\big)$$

Notation. Change of variable $R \mapsto \rho =: \tfrac{1}{2}\ln f'(R)$
Function f and its Legendre transform:

$$w_1(\rho) := f(R) \qquad\qquad w(\rho) := \frac{f(R) - R\,f'(R)}{(f'(R))^2}$$

Field equations of modified gravity in the Einstein metric

$$R^\dagger{}_{\alpha\beta} - 6\,\nabla^\dagger_\alpha\rho\nabla^\dagger_\beta\rho + \tfrac{1}{2}w(\rho)g^\dagger{}_{\alpha\beta} = 8\pi\,e^{-2\rho}\big(T_{\alpha\beta} - \tfrac{1}{2}g^\dagger{}_{\alpha\beta}g^{\dagger\,\alpha'\beta'}\,T_{\alpha'\beta'}\big)$$

THE AUGMENTED CONFORMAL MODEL

‣ At this stage, still third-order and not of a specific PDE type

Augmented formulation

  ‣ relation $e^{2\rho} = f'(R_g)$ *no longer imposed*

  ‣ ρ replaced by a **new independent variable** ϱ

  ‣ algebraic constraint $e^{2\rho} = f'(R_g)$ replaced by the trace equation

  ‣ new notation for the metric $g^{\ddagger}_{\alpha\beta} = e^{2\varrho} g_{\alpha\beta}$

Introduce the tensor field $N^{\ddagger}$ defined by the relation

$$N^{\ddagger}{}_{\alpha\beta} - \frac{1}{2} g^{\ddagger}_{\alpha\beta} \mathrm{Tr}^{\ddagger}(N^{\ddagger}) := e^{2\varrho} \left(R^{\ddagger}{}_{\alpha\beta} - 6 e^{2\varrho} \partial_{\alpha}\varrho \partial_{\beta}\varrho + \frac{1}{2} w(\varrho) g^{\ddagger}{}_{\alpha\beta} \right)$$

‣ Main unknowns: the function ϱ and the metric $g^{\ddagger}_{\alpha\beta}$

‣ A second-order system

- $\widetilde{\Box}_{g^\dagger} := g^{\dagger\alpha'\beta'}\partial_{\alpha'}\partial_{\beta'}$ $\qquad\qquad$ $U(\phi) = \frac{c^2}{2}\phi^2$

- $V = V(\varrho)$ and $W = W(\varrho)$ of quadratic order as $\varrho \to 0$

The augmented conformal formulation of modified gravity in wave gauge

$$\widetilde{\Box}_{g^\dagger}g^\dagger{}_{\alpha\beta} = F_{\alpha\beta}(g^\dagger; \partial g^\dagger, \partial g^\dagger) - 12\partial_\alpha\rho\partial_\beta\rho + V(\rho)g^\dagger{}_{\alpha\beta}$$

$$- 8\pi\left(2e^{-2\rho}\partial_\alpha\phi\partial_\beta\phi + c^2\phi^2 e^{-4\rho}\, g^\dagger{}_{\alpha\beta}\right)$$

$$3\kappa\widetilde{\Box}_{g^\dagger}\rho - \rho = W(\rho) - 8\pi\left(g^{\dagger\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 2c^2\, e^{-2\kappa\rho}\phi^2\right)$$

$$\widetilde{\Box}_{g^\dagger}\phi - c^2\phi = c^2\left(e^{-2\rho} - 1\right)\phi + 2g^{\dagger\alpha\beta}\partial_\alpha\phi\partial_\beta\rho$$

with constraints (propagate from a Cauchy hypersurface)

- $g^{\dagger\alpha\beta}\Gamma^\dagger{}^\lambda_{\alpha\beta} = 0$
- $e^{2\rho} = f'(R_{e^{-2\rho}g^\dagger})$
- Hamiltonian and momentum constraints of modified gravity

NONLINEAR STABILITY PROPERTY

New approach for proving the nonlinear stability of Minkowski spacetime   ArXiv:1511.03324

- ‣ All earlier works concerned vacuum spacetimes or massless scalar fields
  - ‣ Vacuum spacetimes
    - ‣ Christodoulou-Klainerman 1993: null foliation / maximal foliation (fully geometric proof, Bianchi identities, geometry of null cones)
    - ‣ Lindblad-Rodnianski 2010: wave coordinates (almost flat foliation)
  - ‣ Massless models
    - ‣ Bieri-Zipser (2009, weaker decay assumptions), Speck (2014)

- ‣ Cover self-gravitating massive fields
  - ‣ Einstein's gravity theory
  - ‣ $f(R)$-Modified gravity theory
  - ‣ Remark on asymptotically anti-Sitter (AdS) spacetimes:
    - ‣ instabilities are observed ! the effect of gravity is dominant
    - ‣ generic (even arbitrarily small) initial data lead to black hole formation
    - ‣ in AdS spacetime, matter is confined and cannot disperse.

Numerical evidence for stability in asymptotically flat spacetimes

- family of "oscillating soliton stars"
  - suggest a possible instability mechanism within small perturbations of massive fields
- advanced numerical methods necessary to handle the long-time evolution of oscillating soliton stars
  - H. Okawa, V. Cardoso, and P. Pani, Collapse of self-interacting fields in asymptotically flat spacetimes: do self-interactions render Minkowski spacetime unstable?, Phys. Rev., 2014.
    - In a first phase of the evolution, the matter tends to collapse
    - in a next phase below a certain threshold in the mass amplitude, the collapse slows down
    - due to dissipation effects, dispersion becomes the main feature in the evolution of the matter.

Convergence analysis: modified gravity toward Einstein gravity

- Formally when $f(R) \to R$, one has $N_{\alpha\beta} \to G_{\alpha\beta}$.
- *Highly singular:* Einstein equations involve second-order derivatives, while the $f(R)$-field equations contains fourth-order terms.

Taking the trace of the field equations

$$3 \square_g f'(R) + \left(-2f(R) + f'(R)R \right) = 8\pi \, g^{\alpha\beta} T_{\alpha\beta}$$

Evolution equation for the spacetime curvature

- Principal part: $\square R - \frac{1}{3\kappa} R$ in which $\kappa > 0$
- Nonlinear Klein-Gordon equation
- Limit $\kappa \to 0$: spacetime scalar curvature relaxation phenomena passing from a second-order wave equation to an algebraic equation

Theorem 1. Nonlinear stability of Minkowski spacetime of self-gravitating massive fields (PLF-YM 2014 & 2015)

Consider the Einstein-massive field system when the initial data set $(\overline{M} \simeq \mathbb{R}^3, \overline{g}_0, \overline{k}_0, \phi_0, \phi_1)$ satisfies the Einstein constraint equations and is a perturbed hyperboloidal slice.

Then, the corresponding initial value problem

- admits a globally hyperbolic Cauchy development,
- which is foliated by asymptotically hyperbolic hypersurfaces.
- Moreover, this spacetime is future causally geodesically complete and asymptotically approaches Minkowski spacetime.

Theorem 2. Nonlinear stability of Minkowski spacetime in the theory modified gravity (PLF-YM 2016)

Let $(\overline{M} \simeq \mathbb{R}^3, \overline{g}_0, \overline{k}_0, R_0, R_1, \phi_0, \phi_1)$ be an initial data set which satisfies the constraint equations of modified gravity and is a perturbed hyper. slice.

Then, the corresponding initial value problem

- admits a globally hyperbolic Cauchy development,
- which is foliated by asymptotically hyperbolic hypersurfaces.
- Moreover, this spacetime is causally geodesically complete and asymptotically approaches Minkowski spacetime.

SINGULAR LIMIT PROBLEM

- For all $n \leqslant N$, assume $\left| \frac{d^n}{dr^n} \left(f(r) - r - \frac{\kappa}{2} r^2 \right) \right| \lesssim \kappa^{n+2} r^3$

- The convergence property $f(R) \to R$ is equivalent to $\kappa \in [0,1]$ converging to zero.

- E.g., the quadratic action $\int_M \left(R_g + \frac{\kappa}{2}(R_g)^2 + 16\pi L[\phi, g] \right) dV_g$

Theorem 3. The convergence of the modified gravity theory toward Einstein's gravity theory (PLF-YM 2016)

The Cauchy developments of modified gravity in the limit $\kappa \to 0$

    when the nonlinear function $f = f(R)$ arising in the Hilbert-Einstein action tends to the scalar curvature function R

converge (in a sense specified quantitatively below) to those of Einstein's gravity theory.

THE HYPERBOLOIDAL FOLIATION METHOD

- ‣ Scaling vector field does not commute with the Klein-Gordon operator
- ‣ Revisit the vector field method
  - ‣ Hyperboloidal energy based on the Lorentz boosts $L_a := x^a \partial_t + t \partial_a$, only
  - ‣ Weighted norms adapted to the Minkowski geometry
  - ‣ Sharp rates of time decay
    - ‣ Sharp sup-norm estimates for wave and Klein-Gordon operators on curved spaces
    - ‣ Needed to control interaction terms related to the curved geometry
  - ‣ Sobolev inequality on hyperboloids
  - ‣ Hardy-type inequality for the hyperboloidal foliation

Encompass successively broader **classes of coupled systems of wave-Klein-Gordon equations**

- ‣ Nonlinear wave equations with null forms   ArXiv:1411.4910
- ‣ Weak metric interactions + null forms   ArXiv:1411.4910
- ‣ Strong metric interactions + null forms   ArXiv:1507.01143
- ‣ Strong metric interactions + quasi-null forms   ArXiv:1511.03324