Progress on the numerical calculation of the self-force in the time domain

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The problem.

We wish to determine the self-forced motion and field (e.g. energy and angular momentum fluxes) of a particle with scalar charge

$$\Box\psi^{\text{ret}} = -4\pi q \int \delta^{(4)}(x - z(\tau))\, d\tau.$$

2 general approaches:

- Compute enough "geodesic"-based self-forces and then use these to drive the motion of the particle. (Post-processing, fast, accurate self-forces, relies on slow orbit evolution)
- Compute the "true" self-force <u>while</u> simultaneously driving the motion. (Potentially slow and expensive, potentially less accurate self-forces)

Effective source approach.

... is a general approach to self-force and self-consistent orbital evolution that doesn't use any delta functions.

Key ideas

- Compute a regular field, ψ^{R}, such that the self-force is

$$F_\alpha = \nabla_\alpha \psi^{\mathsf{R}}|_{x=z},$$

  where $\psi^{\mathsf{R}} = \psi^{\mathsf{ret}} - \psi^{\mathsf{S}}$, and the Detweiler-Whiting singular field ψ^{S} can be approximated via local expansions: $\psi^{\mathsf{S}} = \tilde{\psi}^{\mathsf{S}} + O(\epsilon^n)$.

- The effective source, S, for the field equation for ψ^{R} is regular at the particle location

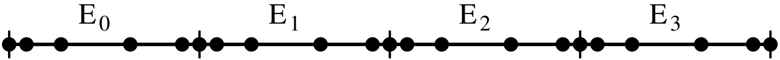
$$\Box\psi^{\mathsf{R}} = \Box\psi^{\mathsf{ret}} - \Box\psi^{\mathsf{S}} = S(x|z,u),$$

  where $\Box\psi^{\mathsf{S}} = -4\pi q \int \delta^{(4)}(x - z(\tau))\, d\tau - S$.

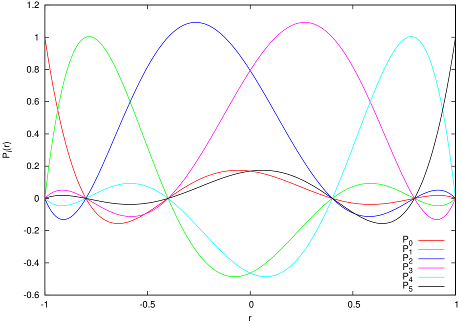
Self-consistent vs. geodesic evolutions.

- One main goal is to compare our self-consistent evolutions with Niels Warburton's geodesic evolutions.
- First attempt: 3+1 multi-patch finite difference code with a C^0 effective source.
- 3+1 accuracy limited by the non-smoothness of the source leading to high frequency noise with 2nd order convergent amplitude.
- Self-consistent evolutions agreed beautifully with geodesic evolutions within the errors (dominated by the noise).
- Next attempt: 3+1 multi-patch finite difference code with a C^2 effective source.
- Geodesic evolution agreed with the C^0 evolutions and the frequency domain result with the noise reduced by more than an order of magnitude.
- However, we found differences between C^2 and C^0 results as soon as the back-reaction was turned on.

Discontinuous Galerkin method.



- Split the domain into N nth order elements.
- Each element contains $n+1$ nodes.
- $u(t,x) \approx \sum_{i=0}^{n} \tilde{u}(t,x_i)P_i(x)$



- The numerical approximation is double valued at all element boundaries.
- Derivatives are approximated by multiplying the state vector in each element by a derivative matrix.
- Neighboring elements are glued together by numerical fluxes based on characteristics.

Code description.

We use a 1+1 dimensional code based on the spherical harmonic decomposition of the scalar wave equation in Schwarzschild tortoise coordinates $r_* = r + 2M\log(r/(2M) - 1)$. The effective source is also decomposed into spherical harmonics.

$$-\frac{\partial^2\psi_{\ell m}}{\partial t^2} + \frac{\partial^2\psi_{\ell m}}{\partial r_*^2} - V_\ell(r)\psi_{\ell m} = S^{\rm eff}_{\ell m}.$$

As $r_* \in [-\infty, \infty]$ we split the domain into three regions. In the inner ($r_* \in [-\infty, T_1]$) and outer ($r_* \in [T_2, \infty]$) regions we introduce new coordinates (τ, ρ) used in Bernuzzi, Nagar & Zenginoğlu (2011).

$$\begin{aligned} t &= \tau + h(\rho) \\ r_* &= \rho/\Omega(\rho) \end{aligned}$$

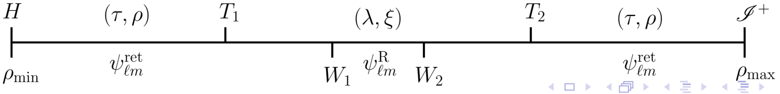
where $h(\rho)$ and $\Omega(\rho)$ are chosen suitably (hyperboloidal layers) in each region to make the inner boundary ($\rho_{\rm min}$) coincide with the horizon H and the outer boundary ($\rho_{\rm max}$) coincide with $\mathscr{I}^+$.

Code description.

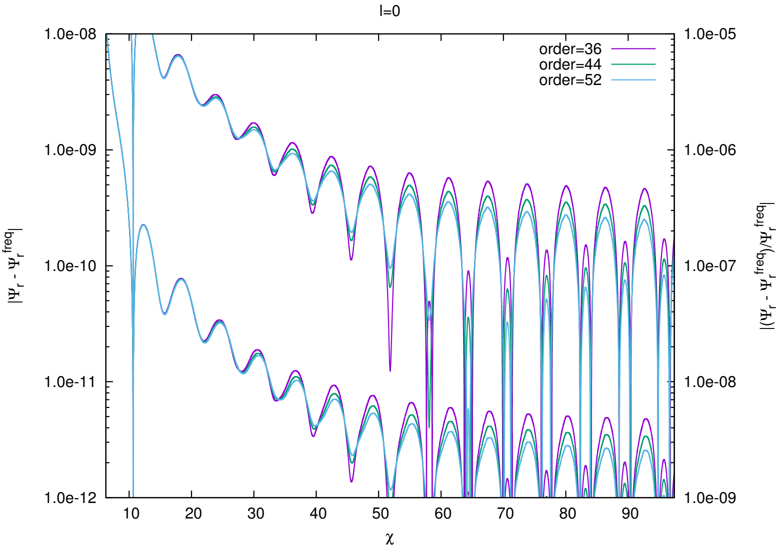
In the middle region ($r_* \in [T_1, T_2]$) we introduce a time dependent coordinate transformation (Field, Hesthaven & Lau, 2009)

$$
\begin{aligned}
t &= \lambda \\
r_* &= T_1 + \frac{r_*^p - T_1}{\xi^p - T_1}(\xi - T_1) + \\
&\quad \frac{(T_2 - r_*^p)(\xi^p - T_1) - (r_*^p - T_1)(T_2 - \xi^p)}{(\xi^p - T_1)(T_2 - \xi^p)(T_2 - T_1)}(\xi - T_1)(\xi - \xi^p)
\end{aligned}
$$

where r_*^p is the time-dependent particle location. This satisfies $r_*(\lambda, T_1) = T_1$, $r_*(\lambda, \xi^p) = r_*^p$, $r_*(\lambda, T_2) = T_2$. In addition we use the world tube approach so that we evolve $\psi_{\ell m}^{\mathrm{R}} = \psi_{\ell m}^{\mathrm{ret}} - \psi_{\ell m}^{\mathrm{S}}$ in the region $r_* \in [W_1, W_2]$, while elsewhere we evolve $\psi_{\ell m}^{\mathrm{ret}}$.
The values of T_1, W_1, W_2 and T_2 are chosen to coincide with element boundaries.



Errors for geodesic eccentric orbit ($e = 0.1, p = 9.9$).



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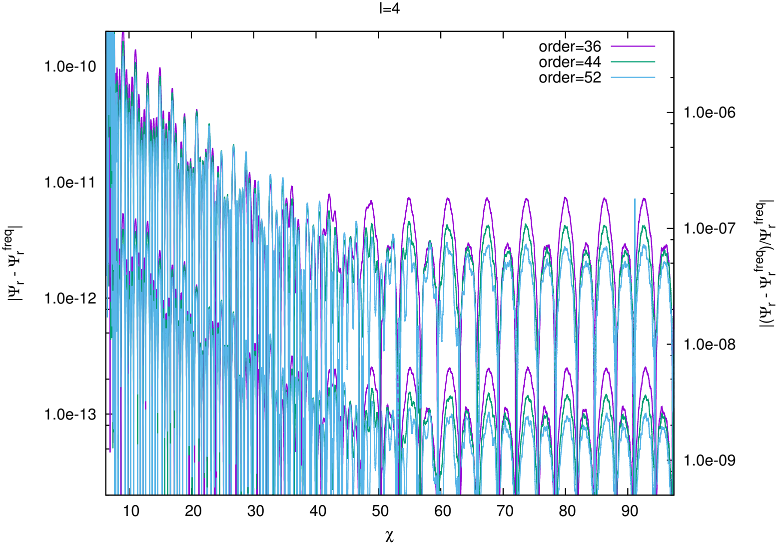

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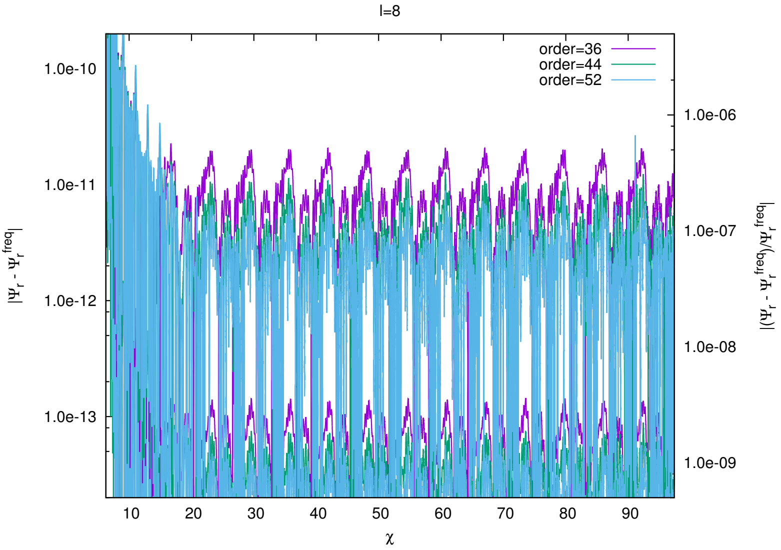

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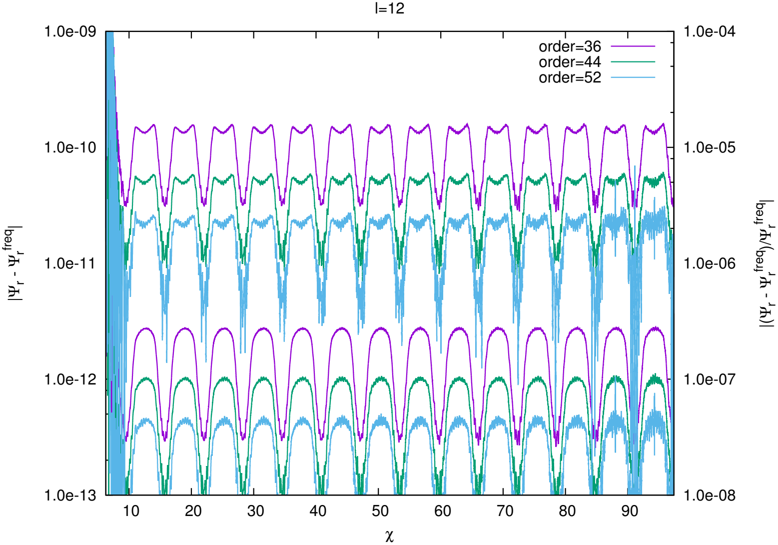

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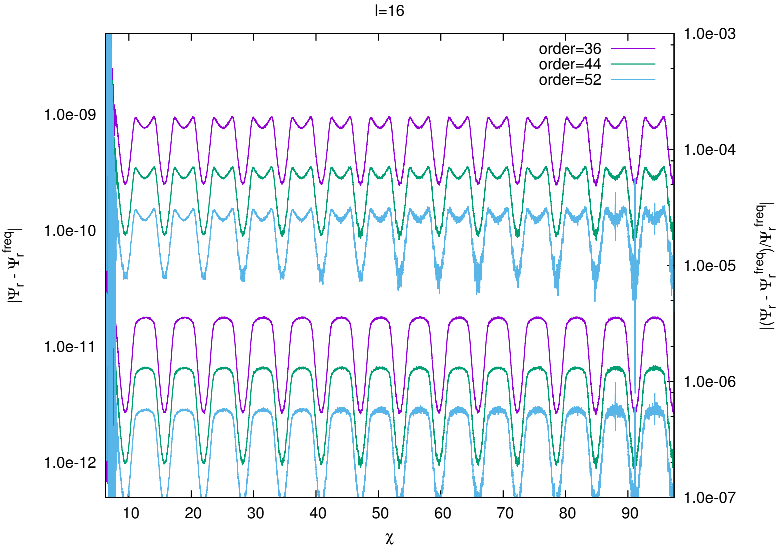

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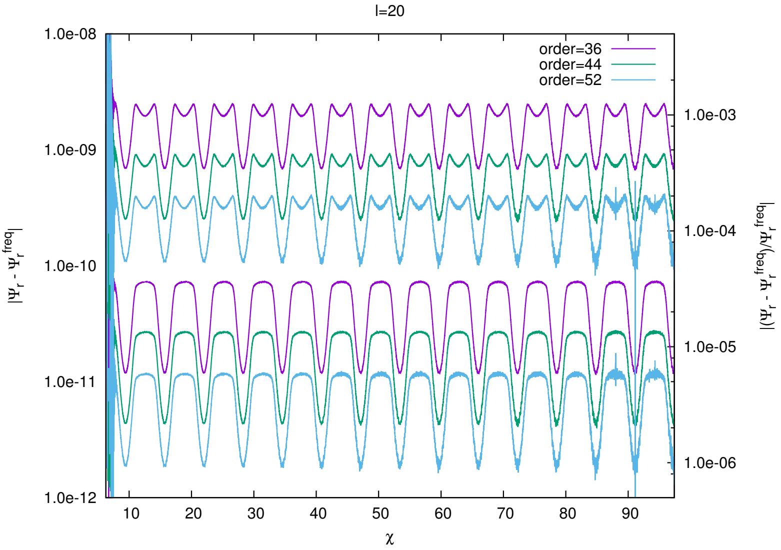
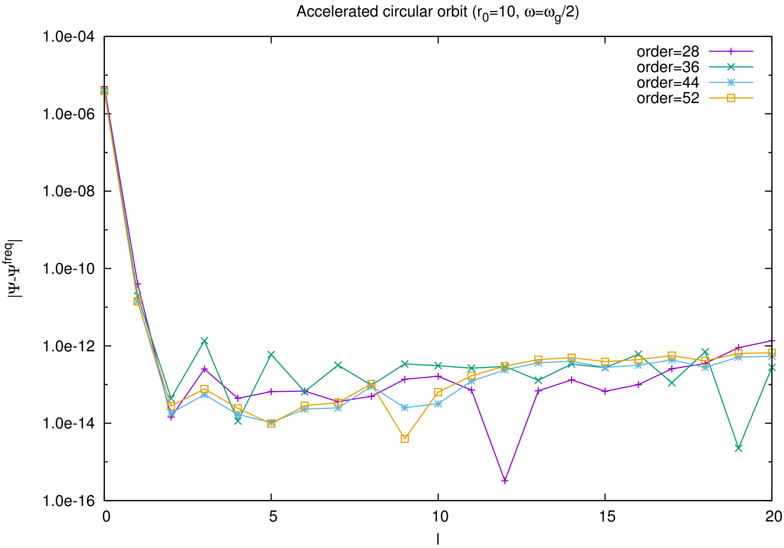


Errors for accelerated circular orbit ($r_0 = 10, \omega = \frac{1}{2}\omega_g$).



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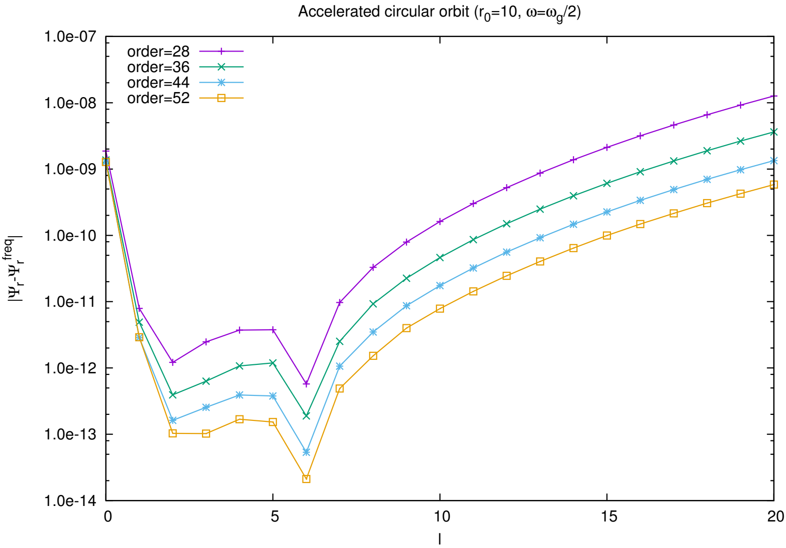

Conclusions and Outlook.

- Discontinuous Galerkin is a powerful numerical method that allows us to overcome the non-smoothness of the effective source.
- The accuracy has been improved and computational cost reduced by at least 2 to 3 orders of magnitude.
- Eccentric geodesic orbits and constant accelerated circular orbits works very well.
- We are working on making accelerated eccentric orbits and non-constant accelerated circular orbits work (very close).
- Self-consistent evolutions are just around the corner.
- We have a finite difference prototype of a coupled mode evolution code for scalar fields in Kerr (to be ported to DG).
- Gravitational perturbation codes (both Lorentz and Regge-Wheeler) are in various stages of development/testing.