Method to compute the stress tensor for quantum fields outside of a black hole that forms from collapse

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Collaborators on various aspects of the project

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- Michael Good, Nazarbayev University, Republic of Kazakhstan
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Why compute $\langle T_{ab} \rangle$?

- Good way to study quantum effects
- Could provide insight into the information issue and the question what is the end point of black hole evaporation?
- Needed to solve the semiclassical backreaction equations

$$G_{ab} = 8\pi \langle T_a b \rangle$$

- Numerical computations in 4D have been done for eternal black holes but not those that form from collapse

Technical difficulties

- Wave equations for the quantum fields are not completely separable

- Renormalization scheme that works for numerical computations must be worked out

Way to Sidestep PDE's

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- that is a solution to the classical Einstein equations
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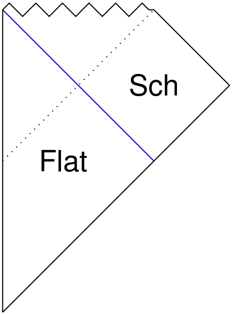
Advantages

- Flat space inside the shell - solutions to mode equation are known
- Schwarzschild spacetime outside - mode equation is separable

Spherically Symmetric Collapsing Null Shell

- Used by Vilkovisky, Fabbri and Navarro-Salas, ... to study aspects of the Hawking effect in 4D



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- Find

$$G^{(1)}(x,x') = \sum_{\ell,m}\int_0^\infty d\omega\{f^{\rm in}_{\omega\ell m}(x)f^{\rm in\,*}_{\omega\ell m}(x') + cc\}$$

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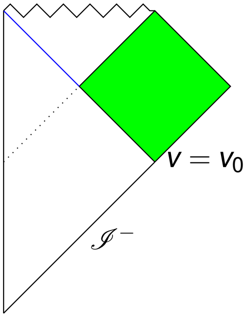
- Inside the shell f^{in} is flat space mode function
- Problem is finding f^{in} outside the shell

f^{in} in Sch region



- Want to compute $\langle T_{ab} \rangle$ in the shaded region

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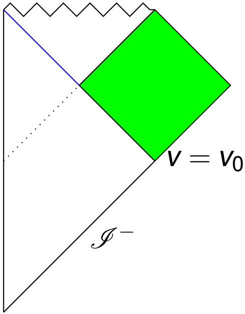
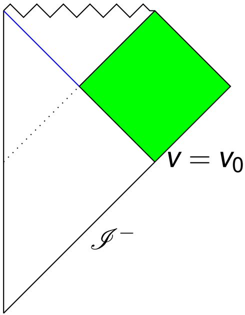


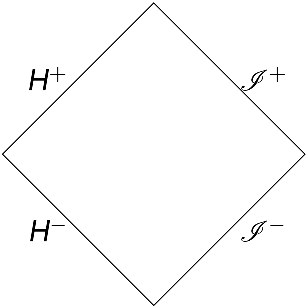
- Want to compute $\langle T_{ab} \rangle$ in the shaded region
- Know f^{in} on $v = t + r = v_0$ and on $\mathscr{I}^-$ for $v > v_0$

$f^{\rm in}$ in Sch region



- Want to compute $\langle T_{ab} \rangle$ in the shaded region
- Know $f^{\rm in}$ on $v = t + r = v_0$ and on $\mathscr{I}^-$ for $v > v_0$
- Expand $f^{\rm in}$ in terms of a complete set of Schwarzschild modes

Pure Schwarzschild Modes



- Complete orthonormal set: f^{H^-} and $f^{\mathscr{I}^-}$.
- On H^-, $f^{H^-} \sim e^{-i\omega(t-r_*)}$
- On $\mathscr{I}^-$, $f^{\mathscr{I}^-} \sim e^{-i\omega(t+r_*)}$

Expansion of the *in* modes

- In pure Schwarzschild spacetime

$$
\begin{aligned}
(f^{\text{in}}_{\omega\ell m})_{\text{Sch}} &= \sum_{\ell',m'}\int_0^\infty d\omega' \Big[A^{\mathscr{I}^-}_{\omega\ell m\omega'\ell'm'} f^{\mathscr{I}^-}_{\omega'\ell'm'} + B^{\mathscr{I}^-}_{\omega\ell m\omega'\ell'm'} (f^{\mathscr{I}^-}_{\omega'\ell'm'})^* \\
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- Modes are orthonormal w.r.t. the scalar product

$$
(\phi_1, \phi_2) = -i \int_\Sigma d\Sigma \sqrt{g_\Sigma} n^a \phi_1 \overleftrightarrow{\partial}_a \phi_2^*
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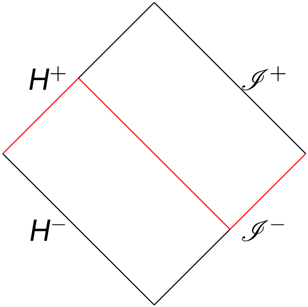
- So

$$
A^{\mathscr{I}^-}_{\omega\ell m\omega'\ell'm'} = \left((f^{\mathrm{in}}_{\omega\ell m})_{\mathrm{Sch}}, f^{\mathscr{I}^-}_{\omega'\ell'm'} \right)
$$

Method



- Key point: Matching is done in pure Schwarzschild

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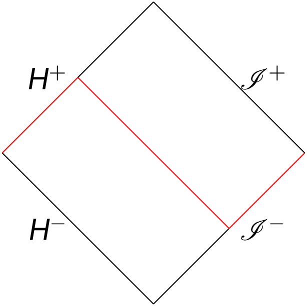

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- Use Cauchy surface in red to compute matching coefficients

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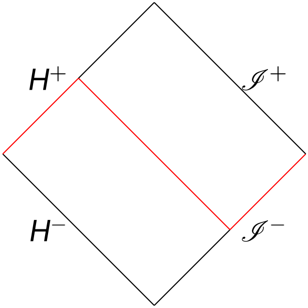

- Key point: Matching is done in pure Schwarzschild
- Use Cauchy surface in red to compute matching coefficients
- Then compute $f^{\text{in}} = \sum_{\ell',m'} \int_0^\infty d\omega' A^{\mathscr{I}^-}_{\omega\ell m\omega'\ell'm'} f^{\mathscr{I}^-}_{\omega'\ell'm'} + \dots$

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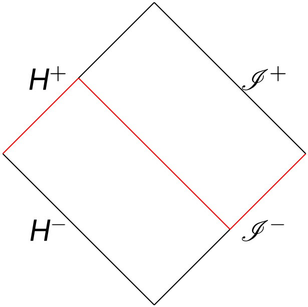

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- Then compute $f^{\rm in} = \sum_{\ell',m'} \int_0^\infty d\omega' A^{\mathscr{I}^-}_{\omega\ell m\omega'\ell'm'} f^{\mathscr{I}^-}_{\omega'\ell'm'} + \dots$
- Then compute $G^{(1)}$ and $\langle T_{ab} \rangle$

Renormalization

Use point splitting - Christensen (1976)

- Adapted for numerical computations in static BH spacetimes - Candelas and Howard (1984), Jensen and Ottewill (1989), Anderson, Hiscock, and Samuel (1995), Levi and Ori (2015)
- More adaptation may be necessary here

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Alternative: Compute difference with Unruh state

$$\langle in|T_{ab}|in\rangle_{\mathrm{ren}} = (\langle in|T_{ab}\, in\rangle_u - \langle U|T_{ab}\, U\rangle_u) + \langle U|T_{ab}\, U\rangle_{\mathrm{ren}}$$

Connection with the Unruh state

$$
\begin{aligned}
(f^{\mathrm{in}}_{\omega\ell m})_{\mathrm{Sch}} &= \sum_{\ell',m'} \int_0^\infty d\omega' \Big[A^{\mathscr{I}^-}_{\omega\ell m\omega'\ell'm'} f^{\mathscr{I}^-}_{\omega'\ell'm'} + B^{\mathscr{I}^-}_{\omega\ell m\omega'\ell'm'} (f^{\mathscr{I}^-}_{\omega'\ell'm'})^* \\
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\end{aligned}
$$

- For the Unruh state, the complete set of modes are: f^K (positive freq. in Kruskal time on H^-) plus the $f^{\mathscr{I}^-}$ modes

$$
f^K_{\omega\ell m} = \sum_{\ell',m'} \int_0^\infty d\omega [\alpha^K_{\omega\ell m,\omega'\ell'm'} f^{H^-}_{\omega'\ell'm'} + \beta^K_{\omega\ell m\omega'\ell'm'} f^{H^-\,*}_{\omega'\ell'm'}]
$$

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$$(f^{\text{in}}_{\omega\ell m})_{\text{Sch}} = \sum_{\ell',m'}\int_0^\infty d\omega' \Big[A^{\mathscr{I}^-}_{\omega\ell m\omega'\ell'm'} f^{\mathscr{I}^-}_{\omega'\ell'm'} + B^{\mathscr{I}^-}_{\omega\ell m\omega'\ell'm'}(f^{\mathscr{I}^-}_{\omega'\ell'm'})^* + A^{H^-}_{\omega\ell m\omega'\ell'm'} f^{H^-}_{\omega'\ell'm'} + B^{H^-}_{\omega\ell m\omega'\ell'm'}(f^{H^-}_{\omega'\ell'm'})^*\Big]$$

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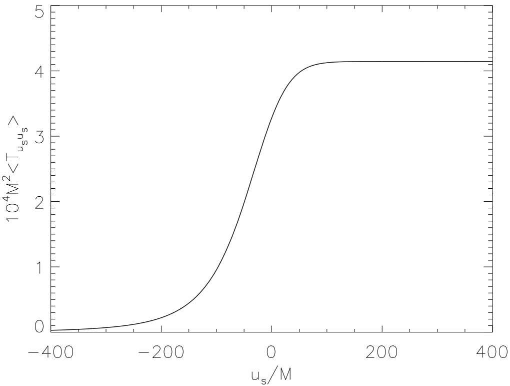
- For $\omega \gg \omega'$ and $\omega'^2 \gg (M/r^3, \ell^2/r^2)$, $A^{H^-} \to -\alpha^K$ and $B^{H^-} \to -\beta^K$

2D Stress Tensor

- Computed by Hiscock (1981)
- Energy flux approaches Unruh value

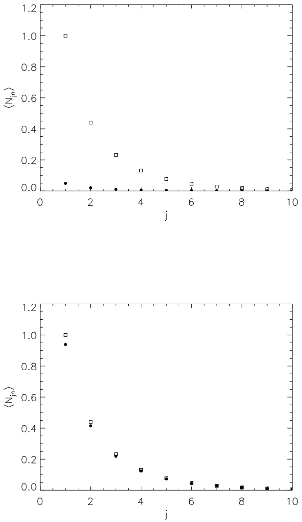


Particle Production for 2D Case

- Computed by Good, Anderson, and Evans
- Use wave packets

$$f_{jn}^{\mathrm{out}} \equiv \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} d\omega e^{2\pi i \omega n/\varepsilon} f_{\omega}^{\mathrm{out}} \, .$$

- *j* corresponds to frequency interval and *n* to time interval
- Match the *in* and *out* modes and use packets to obtain a time dependent spectrum



1.2
1.0
0.8
0.6
0.4
0.2
0.0
$\langle N_{jn} \rangle$
0
2
4
6
8
10
j

Summary

- A method to compute the stress tensor in a spacetime that forms from collapse of a spherically symmetric null shell has been discussed
- A mathematical connection between the matching coefficients for the *in* state and the Unruh state has been found
- Time-dependent spectrum of the produced particles has been computed in 2D showing the approach to a thermal state
- We plan to compute the stress tensor in 4D and also the time-dependent spectrum of the produced particles