On the relationship between the modifications to dynamical equations, and canonical Hamiltonian structures & polymerization

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Introduction

Inverse problem in mechanics: Find the action or the Hamiltonian given the second order differential equation of motion.
(Action can be obtained using Helmholtz conditions; Hamiltonian from symmetry vectors and/or constants of motion,...)

Fundamental questions:

- For a given physical equation of motion whether an action or a Hamiltonian exists, and under what conditions?
- What does the dynamics tell us about the Lagrangian and canonical Hamiltonian structure of the underlying theory?

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Goal: Assuming a specific form of cosmological dynamics, such as which yields generic singularity resolution, find the canonical Hamiltonian/phase space of desired modified gravity theory.

Does demanding a repulsive nature of gravity at high energy scales implies a particular canonically conjugate phase space structure?

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Simple examples:

Gravity repulsive at high spacetime curvature

$$\ddot{a} = -\frac{4\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right) a, \qquad \ddot{a} = -\frac{4\pi G}{3}\rho\left(1 - \frac{\rho^2}{{\rho_c}^2}\right) a, \ \ldots$$

Gravity more attractive at high spacetime curvature

$$\ddot{a} = -\frac{4\pi G}{3}\rho\left(1 + \frac{\rho}{\rho_c}\right) a, \qquad \ddot{a} = -\frac{4\pi G}{3}\rho\left(1 + \frac{\rho^2}{{\rho_c}^2}\right) a, \ \ldots$$

(Here ρ_c a constant to be determined by underlying theory).

Outline of the procedure

Consider a system with position variable q and velocity $\dot{q}$, governed by an equation of motion

$$\ddot{q} = F(q)G(\dot{q})$$

A time independent constant of motion then given by
$C = -\int F dq + \int \dot{q} G^{-1}(\dot{q}) d\dot{q}$ (in this talk: $G(\dot{q}) = 1$).

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A Hamiltonian of the system is C in terms of x_1 and x_2.

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In general, $\{x_1, x_2\} = \mu(x_1, x_2)$. Consistent Hamiltonian evolution requires: $\dot{x}_1 = \mu \frac{\partial \mathcal{H}}{\partial x_2}$, and $\dot{x}_2 = -\mu \frac{\partial \mathcal{H}}{\partial x_1}$

Conjugate momentum: $p = \int \mu^{-1} dx_2$.
Canonical Hamiltonian: $\mathcal{H}(x_1, p)$.

Example: Non-linear restoring force

Consider $\ddot{q} = -q^2$.

Constant of motion: $C = -\int F dq + \int \dot{q} d\dot{q} = \frac{1}{2}\dot{q}^2 + \frac{1}{3}q^3$.

Choose a phase space pair: say, $x_1 = q^2$, and $x_2 = \dot{q}$.

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Consistency of Hamiltonian evolution: $\{x_1, x_2\} = 2x_1^{1/2} = \mu$.

Conjugate momentum of x_1:

$p = \int \mu^{-1} dx_2 = \frac{\dot{q}}{2q}$

Hamiltonian in conjugate variables: $\mathcal{H}(q, p) = 2p^2 q + \frac{1}{3}q^3$

Hamiltonian from Raichaudhuri equation and modifications

Raichaudhuri equation is of the form $\ddot{q} = F(q)$, with q identified with the scale factor a.

Since gravity is a constraint theory, our goal is to find the corresponding Hamiltonian in terms of conjugate phase space variables, whose vanishing gives the physical solutions.

Assumptions:

- The total canonical Hamiltonian of gravity and matter parts is of the form $\mathcal{H}_g + \mathcal{H}_m$
- The matter energy density satisfies the conservation law: $\dot{\rho} + 3H(\rho + P) = 0 \qquad (H = \dot{a}/a)$
- Minimally coupled matter
- No new degrees of freedom when modifications introduced in Raichaudhuri equation

Quadratic repulsive modification to Raichaudhuri equation

$$\ddot{a} = -\frac{4\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right)\, a$$

Choose as phase space variables: $x_1 = a \quad \text{and} \quad x_2 = \dot{a}$.

The constant of motion (which serves as a Hamiltonian):
$C(x_1, x_2) = \frac{x_2^2}{2} - \frac{4\pi G}{3}\frac{\mathcal{H}_m}{a^3}\left(1 - \frac{1}{4\rho_c}\frac{\mathcal{H}_m}{a^3}\right)x_1^2$

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But this Hamiltonian not in the form $\mathcal{H} = \mathcal{H}_g + \mathcal{H}_m$.

However, it can be expressed as:
$\mathcal{H} \approx -\frac{3x_1^3}{8\pi G\alpha^2}\,\frac{1}{2}(1 \pm \sqrt{1 - 4\alpha^2 H^2}) + \mathcal{H}_m;\;\; \alpha^2 := \frac{3}{8\pi G\rho_c}$

Both roots necessary to capture complete dynamics for the entire allowed range of energy density.

Form of $\mathcal{H}$ implies, convenient to switch $x_1 \to a^3$ and $x_2 \to H$.

For negative root, consistent Hamiltonian evolution requires $\{x_1, x_2\} = \mu^- = -\frac{1}{2}\sqrt{1 - 4\alpha^2 x_2^2}$

Conjugate momentum: $p^- = -\beta^{-1}\sin^{-1}(2\alpha x_2)$ $(\beta = 8\pi G\alpha)$
Corresponding gravity part of the Hamiltonian:
$\mathcal{H}_g^- = -\frac{3x_1}{16\pi G\alpha^2}\left(1 - \cos(\beta p^-)\right)$

Similarly for the positive root:
$\mathcal{H}_g^+ = -\frac{3x_1}{16\pi G\alpha^2}\left(1 + \cos(\beta p^+)\right)$ with $p^+ = \beta^{-1}\sin^{-1}(2\alpha x_2)$

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p^+ and p^- do not belong to same range for any H. Hamiltonian can be written in terms of canonical momentum p, defined via p^+ and p^-, covering the entire allowed range of H:

$\mathcal{H} = -\frac{3x_1}{2\alpha^2}\left(1 - \cos(\beta p)\right) + \mathcal{H}_m$
(same as the effective Hamiltonian of spatially flat isotropic LQC!)

Polymerization appears naturally demanding repulsive gravity without any inputs from LQG.

Cubic repulsive modification to Raichaudhuri equation

$$\ddot{a} = -\frac{4\pi G}{3}\rho\left(1 - \frac{\rho^2}{{\rho_c}^2}\right) a$$

Leads to a much more non-trivial canonical phase space. Out of three roots, one unphysical.

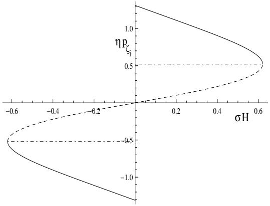
Two conjugate momenta in different ranges (as in quadratic case):

$$p_{\zeta_1} = \frac{1}{4\pi G}\frac{3^{1/4}}{5\sqrt{2}\sigma}\left[\chi^{-1}\sqrt{1+e^{2i\cos^{-1}(-\chi^2)}}e^{-\frac{i}{3}\cos^{-1}(-\chi^2)}\left[-e^{\frac{2i}{3}\cos^{-1}(-\chi^2)}\right.\right.$$
$$\left.\left.\times\,{}_2F_1\left(\tfrac{5}{12},\tfrac{1}{2};\tfrac{17}{12};e^{-2i\cos^{-1}(-\chi^2)}\right)+5\,{}_2F_1\left(\tfrac{1}{12},\tfrac{1}{2};\tfrac{13}{12};e^{-2i\cos^{-1}(-\chi^2)}\right)\right]\right]$$

$$\sigma = (3/(8\sqrt{7}\pi G\rho_c))^{1/2},\ \eta = 4\pi G\sigma,\ \chi^2 = \frac{3\sqrt{3}\sigma^2 H^2}{2}$$

A "generalized" polymerized canonical phase space emerges.

Period of oscillation is 3/2 times smaller than the period in quadratic repulsive case.



What about attractive modifications?

$\ddot{a} = -\frac{4\pi G}{3}\rho\left(1 + \frac{\rho^2}{\rho_c^2}\right)a$ (leads to only one physical root).

Conjugate momentum to a^3: $p = -\beta^{-1}\sinh^{-1}(2\alpha H)$

Canonical Hamiltonian: $\mathcal{H} = -\frac{3a^3}{8\pi G\alpha^2}\sinh^2(\beta p/2) + \mathcal{H}_m$

(Links with brane-world scenarios, complexified connection)

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No polymerization appears when gravity is attractive at high curvature scales.

Summary

- Using simple assumptions, our procedure allows constructing canonical phase space structure, Hamiltonian and action from desired cosmological dynamics.
- It seems that repulsive nature of gravity at high spacetime curvature has links with polymerization. Polymerized phase space appears explicitely for certain cases (including in presence of spatial curvature and anisotropies **(Work in progress with S K Soni)**). Generic resolution of strong singularities in these cases.
- For modified gravity scenarios, which make gravity more attractive at high curvature scales, no polymerization appears. Problem of singularities worse in these scenarios.
- Does singularity resolution implies a polymerized version of gravity at high curvature scales, and naturally lead us towards loop quantum gravity?