Goldberg-Sachs and the alignment of Einstein-Maxwell fields

Norbert Van den Bergh

Department of Mathematical Analysis
University of Gent
Belgium

21st International Conference on
General Relativity and Gravitation - 2016

Einstein-Maxwell + Λ-term:

$$\boxed{R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}}$$

- "aligned case": at least one PND of $\boldsymbol{F}$ is parallel to a Debever[1]-Penrose vector
- "doubly aligned case": both real PND's of $\boldsymbol{F}$ are parallel to a Debever-Penrose vector
- "non-aligned case": no PND's of $\boldsymbol{F}$ are parallel to a Debever-Penrose vector

[1] Westbridge 25.11.1915–Brussels 11.5.1998

Goldberg-Sachs theorem (1962):

$$\mathcal{M} \text{ (vacuum) algebraically special} \iff \mathcal{M} \text{ contains a shear-free geodesic null congruence}$$

generalisation :

$$\boldsymbol{k} \text{ shear-free, geodesic PND of } \boldsymbol{F} \implies \boldsymbol{k} \text{ multiple Debever-Penrose vector}$$

reverse?

Kundt-Trümper (1962:)

$$\boldsymbol{k} \text{ PND of a non-null } \boldsymbol{F} \text{ and multiple Debever-Penrose vector}$$
$$\Longrightarrow$$
$$\kappa(3\Psi_2 - 2|\Phi_1|^2) = 0 \text{ and } \sigma(3\Psi_2 + 2|\Phi_1|^2) = 0$$

hence

$$\boxed{|\kappa|^2 + |\sigma|^2 \neq 0 \Longrightarrow \text{ type II or D with } \frac{3}{2}\Psi_2 = \pm|\Phi_1|^2}$$

Doubly aligned type D

- $\frac{3}{2}\Psi_2 \neq \pm|\Phi_1|^2 \Longrightarrow \kappa = \nu = \sigma = \lambda = 0 \Longrightarrow$ 'class $\mathcal{D}$' (Debever-McLenaghan 1981, Plebański Demiański 1976)

- $\frac{3}{2}\Psi_2 = |\Phi_1|^2$ $(\sigma = 0) \Longrightarrow$ (Plebański-Hacyan 1979) both $\boldsymbol{k}$ and $\boldsymbol{\ell}$ are shear-free, resp. non-geodesic and geodesic and "double Kundt" $(\rho = \mu = 0)$; $\Lambda < 0$

- $\frac{3}{2}\Psi_2 = -|\Phi_1|^2$ $(\kappa = 0) \Longrightarrow$ (García-Plebański 1982) both $\boldsymbol{k}$ and $\boldsymbol{\ell}$ are geodesic, shearing and twisting, but non-expanding $(\Re\rho = \Re\mu = 0)$; $\Lambda < 0$

$$\boxed{\text{What happens in the non-doubly aligned case?}}$$

[VdB, arXiv:1605.05830]

Aligned multiple D-P vector with $|\kappa|^2 + |\sigma|^2 \neq 0$

$$|\kappa|^2 + |\sigma|^2 \neq 0$$

1. Theorem:
   multiple Debever-Penrose vector with $\sigma \neq 0$ and PND of non-null $\boldsymbol{F} \Longrightarrow \Lambda < 0 \Longrightarrow$ García-Plebański
   (this corrects and generalizes Kozarzewski[Acta Phys. Pol. 27, 775, 1965]: $\rho = \pm i|\sigma|$ cannot easily be dismissed).

2. Theorem:
   multiple Debever-Penrose vector with $\kappa \neq 0$:
   $\rho = 0$ (Kundt sub-family) $\Longrightarrow \Lambda < 0 \Longrightarrow$ Plebański-Hacyan
   $\rho \neq 0$: open problem!

Multiple Debever-Penrose vector with $\kappa = \sigma = 0$

3. Theorem:

$$\boldsymbol{k} \text{ multiple Debever-Penrose vector with } \kappa = \sigma = 0 \text{ and}$$
$$\boldsymbol{k} \text{ no PND of } \boldsymbol{F} \Longrightarrow \boldsymbol{F} \text{ non-null and } R = 0$$

Choosing $\boldsymbol{\ell}$ such that $\Phi_1 = 0$:

- $\pi = 0 \Longrightarrow$ Griffiths 1986 (II)
  - $\tau = 0 \Longrightarrow$ Griffiths 1983 (II)
  - $\mu = 0 \Longrightarrow$ Cahen-Spelkens 1967 (III)
  - $\rho\nu = \tau\lambda \Longrightarrow$ Cahen-Leroy 1966 (N)
- $\pi \neq 0 \Longrightarrow$?

4. Corollary:

type D with both Debever-Penrose vectors $\boldsymbol{k}$ and $\boldsymbol{\ell}$
geodesic and shear-free ($\kappa = \sigma = \lambda = \nu = 0$)

- $\Lambda \neq 0 \Longrightarrow$ doubly aligned $\Longrightarrow$ class $\mathcal{D}$
- $\Lambda = 0 \Longrightarrow$?

(Geroch, Held and Penrose, JMP 1973)

$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4 \equiv \boldsymbol{m}, \bar{\boldsymbol{m}}, \boldsymbol{\ell}, \boldsymbol{k}$ with $\boldsymbol{k} \cdot \boldsymbol{\ell} = -1, \boldsymbol{m} \cdot \bar{\boldsymbol{m}} = 1$

Under boosts

$$\boldsymbol{k} \to A\boldsymbol{k}, \ \boldsymbol{\ell} \to A^{-1}\boldsymbol{\ell}$$

and spatial rotations

$$\boldsymbol{m} \to e^{i\theta}\boldsymbol{m}$$

well-weighted variables η of weight $[p, q]$ transform as

$$\eta \to A^{\frac{p+q}{2}} e^{i\frac{p-q}{2}\theta} \eta$$

(η has *boost-weight* $= \frac{p+q}{2}$ and *spin-weight* $= \frac{p-q}{2}$).

Basic variables:

$$\kappa = \Gamma_{414}, \quad \tau = \Gamma_{413}, \quad \sigma = \Gamma_{411}, \quad \rho = \Gamma_{412},$$
$$\nu = \Gamma_{233}, \quad \pi = \Gamma_{234}, \quad \lambda = \Gamma_{232}, \quad \mu = \Gamma_{231},$$

$(\Gamma_{abc} = -\Gamma_{bac} \equiv \boldsymbol{e}_a \nabla_c (\boldsymbol{e}_b))$,

$$\Phi_{00}, \Phi_{22}, \Phi_{01}, \Phi_{12}, \Phi_{02}, \Phi_{11},$$
$$R, \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4.$$

$\alpha, \beta, \epsilon, \gamma$ get absorbed in $þ, þ', ð, ð'$:

$$\begin{aligned} þ\eta &= (D - p\epsilon - q\bar{\epsilon})\eta \\ þ'\eta &= (\Delta - p\gamma - q\bar{\gamma})\eta \\ ð\eta &= (\delta - p\beta - q\bar{\alpha})\eta \\ ð'\eta &= (\bar{\delta} - p\alpha - q\bar{\beta})\eta \end{aligned}$$

Symmetry transformations:

- complex conjugation
- prime transformation:

$$\boldsymbol{k} \leftrightarrow \boldsymbol{\ell}, \boldsymbol{m} \leftrightarrow \bar{\boldsymbol{m}},$$

$$\kappa \leftrightarrow -\nu, \quad \tau \leftrightarrow -\pi, \quad \sigma \leftrightarrow -\lambda, \quad \rho \leftrightarrow -\mu,$$
$$\Phi_{ij} \leftrightarrow \Phi_{2-i\,2-j}, \quad \Psi_i \leftrightarrow \Psi_{4-i}$$

- Sachs transformation:

$$\boldsymbol{k} \rightarrow \boldsymbol{m}, \boldsymbol{\ell} \rightarrow \bar{\boldsymbol{m}}, \boldsymbol{m} \rightarrow \bar{\boldsymbol{k}}, \bar{\boldsymbol{m}} \rightarrow \boldsymbol{\ell}$$

$$\rho^* = \tau, \tau^* = -\rho, \rho'^* = -\tau', \tau'^* = \rho'$$

Basic equations:

- 12 complex Ricci equations

$$
\begin{aligned}
Þ\tau - Þ'\kappa &= (\tau - \bar{\tau}')\rho + (\bar{\tau} - \tau')\sigma + \Phi_{01} + \Psi_1, \\
ð\rho - ð'\sigma &= (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa + \Phi_{01} - \Psi_1, \\
Þ\sigma - ð\kappa &= (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0, \\
Þ\rho - ð'\kappa &= \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa\tau' + \Phi_{00}, \\
Þ'\sigma - ð\tau &= \sigma\rho' - \bar{\lambda}\rho - \tau^2 + \kappa\bar{\nu} - \Phi_{02}, \\
Þ'\rho - ð'\tau &= \rho\bar{\rho}' - \lambda\sigma - \tau\bar{\tau} + \kappa\nu - \Psi_2 - \frac{1}{12}R
\end{aligned}
$$

- Maxwell equations

$$
\begin{aligned}
Þ\Phi_1 - ð'\Phi_0 &= \pi\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 \\
Þ\Phi_2 - ð'\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + \rho\Phi_2
\end{aligned}
$$

GHP formalism

- 9 complex + 2 real Bianchi equations
- commutator relations

Define extension variables U, V:

$$Þ'\Phi_0 = U,\ ð'\Phi_0 = V$$

Maxwell equations

$$Þ\Phi_1 = \pi\Phi_0 + 2\rho\Phi_1 + V$$
$$ð\Phi_1 = \mu\Phi_0 + 2\tau\Phi_1 + U$$

GHP equations

$$
\begin{aligned}
\text{Þ}\rho &= \rho^2 + |\Phi_0|^2 \\
\text{Þ}\tau &= \rho(\tau + \bar{\pi}) + \Phi_0\overline{\Phi_1} \\
\text{Þ}\mu - \eth\pi &= \pi\bar{\pi} + \mu\bar{\rho} + \frac{1}{12}R + \Psi_2 \\
\eth\rho &= \tau(\rho - \bar{\rho}) + \Phi_0\overline{\Phi_1} \\
\eth\tau &= \tau^2 + \Phi_0\overline{\Phi_2}
\end{aligned}
$$

Bianchi equations $\Longrightarrow$

$$\begin{aligned} Þ\Psi_2 &= 3\rho\Psi_2 + 2\Phi_1(\rho\overline{\Phi_1} - \tau\overline{\Phi_0}) - U\overline{\Phi_0} + V\overline{\Phi_1} \\ ð\Psi_2 &= 3\tau\Psi_2 + 2\Phi_1(\rho\overline{\Phi_2} - \tau\overline{\Phi_1}) - U\overline{\Phi_1} + V\overline{\Phi_2} \end{aligned}$$

and

$$\begin{aligned} \overline{\Phi_1}Þ\Phi_0 - \overline{\Phi_0}ð\Phi_0 &= 0 \\ \overline{\Phi_2}Þ\Phi_0 - \overline{\Phi_1}ð\Phi_0 &= 0 \end{aligned}$$

hence

$$Þ\Phi_0 = ð\Phi_0 = 0$$

Integrability conditions $\Longrightarrow$

$$
\begin{aligned}
\eth V &= (\rho - \bar{\rho})U - 2\Phi_0(|\Phi_1|^2 - \Psi_2 + \rho\mu + \frac{1}{24}R) \\
\text{Þ} V &= \rho V - 2\Phi_0(\rho\pi + \Phi_1\overline{\Phi_0}) \\
\eth U &= \tau U - 2\Phi_0(\mu\tau + \Phi_1\overline{\Phi_2}) \\
\text{Þ} U &= (\bar{\pi} - 2\tau)V + 3\rho U - \Phi_0(2\tau\pi + 2|\Phi_1|^2 - \Psi_2 + \frac{1}{6}R)
\end{aligned}
$$

after which $[\text{Þ}', \text{Þ}]\Phi_0$ gives

$$\Phi_0(\Psi_2 - \frac{1}{12}R) + \rho U - \tau V = 0$$

the derivatives of which imply

$$\tau\Phi_0 R = \rho\Phi_0 R = 0$$