Scattering for 2+1 Equivariant Einstein wave map system

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Wave Equation, Wave Maps

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- Satisfies the (strong) null condition
- H^1 data is critical.

The Cauchy Problem

Let Σ be the initial Cauchy surface and $\mathbf{X}$ be its unit normal, then the Cauchy problem of wave maps is the following

$$\left.\begin{array}{rcl} \Box_g U^i + {}^{(h)}\Gamma^i_{jk} g^{\alpha\beta} \partial_\alpha U^j \partial_\beta U^k & = & 0 \text{ on } M \\ U|_\Sigma & = & U_0 \\ \mathbf{T}(U)|_\Sigma & = & U_1 \end{array}\right\} \tag{1}$$

such that

$$\begin{aligned} U_0 \; : \; \Sigma &\to N \\ p &\to U_0(p) \end{aligned}$$

and

$$\begin{aligned} U_1 \; : \; \Sigma &\to T_{U_0} N \\ p &\to T_{U_0(p)} N \end{aligned}$$

for $p \in \Sigma$.

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- Christodoulou, Shatah, Tahvildar-Zadeh, Tao, Klainerman, Machedon, Selberg, Rodnianski, Raphael, Tataru, Sterbenz, Struwe, Nahmoud, Stefanov, Uhlenbeck, Merle, Kenig, Krieger, Schlag, Lawrie etc.

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- **R** Ricci tensor, **E** Einstein tensor of (M, g)
- Dynamical background (M, g)
- John Wheeler " Spacetime tells matter how to move; matter tells spacetime how to curve "

Cauchy Problem of Critical Einstein Equivariant Wave Map System

$$\left.\begin{array}{rcll} \mathbf{E}_{\mu\nu} & = & \alpha \mathbf{T}_{\mu\nu} & \text{on } M^{2+1} \\ \Box_g u & = & \frac{k^2 f_u(u) f(u)}{r^2} & \text{on } M^{2+1} \\ U|_\Sigma & = & U_0 & \\ T(U)|_\Sigma & = & U_1 & \end{array}\right\} \tag{4}$$

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Initial data $(\Sigma, q, \mathbf{K}, U_0, U_1)$ smooth, compactly supported and satisfying constraint equations.

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Initial data $(\Sigma, q, \mathbf{K}, U_0, U_1)$ smooth, compactly supported and satisfying constraint equations. In what follows assume k (homotopy degree) $= 1$

2+1 EEWM System

In double null-coordinates:

$$
\begin{aligned}
r^{-1}(2\partial_\xi Z \partial_\xi r - \partial_\xi^2 r) =& \partial_\xi u \, \partial_\xi u && (5)\\
r^{-1}\partial_{\xi\eta}^2 r =& \frac{e^{2Z}}{4}\frac{f^2(u)}{r^2} && (6)\\
r^{-1}(2\partial_\eta Z \partial_\eta r - \partial_\eta^2 r) =& \partial_\eta u \, \partial_\eta u, && (7)\\
-4r^2 \, e^{-2Z}\partial_{\xi\eta}^2 Z =& \frac{r^2}{2}\left(4e^{-2Z}\partial_\eta u \partial_\xi u + \frac{f^2(u)}{r^2}\right) && (8)\\
\Box_{g(u)} u =& \frac{f_u(u) f(u)}{r^2}. && (9)
\end{aligned}
$$

Global Existence for small data

Theorem

Suppose (M, g, U) is the globally hyperbolic, maximal development of the initial data of the 2+1 equivariant Einsten-wave map system (Σ_0, q_0, K_0) with $E_0 < \epsilon^2$ for ϵ sufficiently small, with

$$\int_0^u f(s)ds \to \infty \quad as \quad u \to \infty$$

then (M, g, U) is geodesically complete with global regularity.

Steps in the proof

Joint with Andersson and Szeftel

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- Global regularity for u follows

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- Morawetz esimate 1:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^4} \frac{1}{|x|^3} v^2 \bar{\mu}_{\check{g}} \leq \|v_0\|_{\dot{H}^1(\mathbb{R})^4} + \|v_1\|_{L^2(\mathbb{R})^4} \tag{10}$$

- Morawetz estimate 2:

$$\int_{\mathbb{R}} \int_{|x|<\rho} \check{e} \bar{\mu}_{\check{g}} \leq \rho \left(\|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} \right). \tag{11}$$

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$$\|v\|_X \leq \|v_0\|_{\dot{H}^1(\mathbb{R}^4)} + \|v_1\|_{L^2(\mathbb{R}^4)} + \|F\|_Y. \tag{12}$$

- Scattering:

$$\|v\|_{L^2 L^8(\mathbb{R}\times\mathbb{R}^4)} < \infty$$

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- Super-critical EWM system.

Thank you, glad to be here