Lie point symmetries, vacuum Einstein equations, and Ricci solitons

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Gravity, Einstein Equations and Exact Solutions

- General Relativity is considered the most accurate theory describing gravity till today. It is a geometric theory and is described by Einstein equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}$, a set of highly non-linear 2nd order partial differential equations (PDEs) in $g_{\mu\nu}$'s.
- The most nonlinear theory of nature – hard to solve analytically or numerically even with simple matter content except without strong simplifying assumptions.
- Exact solutions help us understand the non-linearities of EE and are harder to find. There are many schemes, techniques and methods to generate, classify, study them: symmetry assumptions, algebraic classification, generation techniques. See Stephani *et al*.

Solution Generation Technique

- The difficulty of directly integrating Einstein's equations has led to many solution-generation techniques in which one obtains a solution, or a family of solutions, from a "seed" solution, of the same system or a different system.
- A common idea is to use symmetries present in the seed solution.
- Buchdahl (1954) showed how to obtain a Ricci-flat solution from another in the presence of a hypersurface-orthogonal Killing vector field.
- Ehlers (1957) showed how one could obtain a stationary axisymmetric metric starting from any static metric.
- Geroch (1972) showed that one can use the two commuting Killing vector fields of any stationary axisymmetric metric to obtain an infinite-parameter family of solutions.

Solution Generation Technique

- Following the discovery of Tomimatsu–Sato solutions (1972) stationary axisymmetric systems were vigorously studied, aided by techniques developed in other systems of PDEs (various Bäcklund and other transformations, inverse-scattering methods etc.).
- Many sophisticated general results and specific solutions were obtained for stationary axisymmetric systems including the Einstein–Maxwell system.
- However, applying those results to obtain explicit solutions, of the same system or another, often involves solving an associated set of equations and performing a good number of mathematical steps. One cannot usually simply write down a new solution starting from a seed solution.

Rest of the Talk

The rest of the talk will present:

- An explicit one-parameter Lie point symmetry of the four-dimensional vacuum Einstein equations with two commuting hypersurface-orthogonal Killing vector fields.
- How this symmetry enables one to construct particular one-parameter extended families of axisymmetric static solutions and cylindrical gravitational wave solutions from old ones, in a simpler way than most solution-generation techniques, including the prescription given by Ernst for this system.
- Also how this enables us to construct new steady Ricci solitons (self-similar solutions of the Ricci flow) by exploiting a correspondence between static solutions of Einstein's equations and Ricci solitons.

EEs with two commuting hypersur-orth KVF in Vacuum

- In Lorentzian signature these include axially symmetric static solutions and (Einstein–Rosen) cylindrical gravitational waves.
- The first one is best described in Weyl coordinates as

$$ds^2 = -e^{2u(\rho,z)}dt^2 + e^{-2u(\rho,z)}\left[e^{2k(\rho,z)}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right] \quad (1)$$

where $u(\rho,z)$ and $k(\rho,z)$ satisfy the following three equations:

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0, \quad (2)$$

$$\frac{\partial k}{\partial \rho} = \rho\left[\left(\frac{\partial u}{\partial \rho}\right)^2 - \left(\frac{\partial u}{\partial z}\right)^2\right], \quad (3)$$

$$\frac{\partial k}{\partial z} = 2\rho\frac{\partial u}{\partial \rho}\frac{\partial u}{\partial z}. \quad (4)$$

- The ER cylindrical gravitational wave system can be obtained by $z \to i\,t$ and $t \to iz$ will not be separated for discussion.

Axially Symmetric Static Solutions in Vacuum

- A "solution" refers to a pair (u, k) solving (2)-(4). Equation (2) is just the axially symmetric Laplace equation in cylindrical coordinates in an auxiliary three-dimensional Euclidean space. For any (harmonic) function $u(z, \rho)$ solving (2), $k(z, \rho)$ is uniquely determined and found by integrating (3) and (4), which reflect the nonlinearities of the Einstein equations. No distinction is made between solutions in which u and/or k differ by additive constants. Examples:
- Flat space: $u = k = 0$ (Cartesian); $u = \ln\rho, k = \ln\rho$ (Rindler); $u = \frac{1}{2}\ln(\sqrt{\rho^2 + z^2} + z)$, $k = \frac{1}{2}\ln\left(\frac{\sqrt{\rho^2+z^2}+z}{2\sqrt{\rho^2+z^2}}\right)$ (Gautreau-Hoffman).
- Weyl solutions, asymptotically flat :
  $u = -\sum_{n=0}^{\infty} a_n r^{-(n+1)} P_n(\cos\theta)$ and
  $k = -\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_l a_m \frac{(l+1)(m+1)}{(l+m+2)} \frac{(P_l P_m - P_{l+1}P_{m+1})}{r^{l+m+2}}$
  (where $\rho = r\sin\theta$ and $z = r\cos\theta$).
- And many more, (see, for example, Stephani *et al*).

Axially Symmetric Static Solutions in Vacuum

Interplay between linearity and nonlinearity:

- Linearity of (2) implies, if (u_1, k_1) and (u_2, k_2) are two solutions, $u = c_1 u_1 + c_2 u_2$ is a solution of (2).
- However, the nonlinearity of (3) and (4) prevents one from obtaining a standard prescription for k in terms of the four quantities $\{u_1, u_2, k_1, k_2\}$.
- One has to compute the line integral of (3)-(4) (or some equivalent set of differential equations) starting with $u = c_1 u_1 + c_2 u_2$, which is no different from the basic problem of solving (3)-(4) for a given u.

Axially Symmetric Static Solutions in Vacuum

- Question: Given an arbitrary solution (u_0, k_0) can one generate another solution by some simpler means without solving the full set?
- Ernst (1978, 1979) gave a method by which one can obtain a new solution $(u_0 + cz, k_0 + cF - \frac{c^2}{2}\rho^2)$ from a given solution (u_0, k_0) provided the real function F satisfies the following (simpler) differential equation

$$\nabla F = 2i\rho\nabla u_0, \tag{5}$$

  where, $\nabla = \partial_\rho + i\partial_z$. (In Ernst's paper, (5) is misprinted and an auxiliary function L is introduced which can be dispensed with.)
- Ernst's method superposes a multiple of the simple cylindrical solution with $u = z$ on (u_0, k_0). Ernst applied this to obtain a one-parameter generalization of the C-metric, and Kerns and Wild similarly obtained a generalization of the Schwarzschild metric. For these one has to solve (5) starting with the seed's u_0, the difficulty of which depends on the functional form of u_0.

Axially Symmetric Static Solutions in Vacuum

Are there further ways of producing new solutions from old without solving the field equations or an equivalent set of equations? This leads one to look for explicit symmetries of the system.

- One immediately finds that the following transformation is a symmetry

$$(u_0, k_0) \rightarrow (\beta u_0, \beta^2 k_0). \tag{6}$$

  Thus for any arbitrary solution (u_0, k_0) there is a (non-equivalent) solution $(\beta u_0, \beta^2 k_0)$ for $\beta \in (-\infty, \infty)$. This has been recently used to generate new solutions and study their properties (Chng, Mann, Stelea, PRD **74**, 2006).

- This is a scaling symmetry and does not mix dependent and independent variables. Simple.

Axially Symmetric Static Solutions in Vacuum

The following mixes variables in a nontrivial way (Akbar, MacCallum, PRD **92**, 2015):

Theorem: For $\alpha \in (-\infty, \infty)$, the transformation

$$(u_0, k_0) \to (u_0 + \alpha \ln \rho, k_0 + 2\alpha u_0 + \alpha^2 \ln \rho), \tag{7}$$

leaves the system (2)-(4) invariant. In other words, for every static axially symmetric vacuum solution of the Einstein equations

$$ds^2 = \pm e^{2u_0(\rho,z)} dt^2 + e^{-2u_0(\rho,z)} \left[e^{2k_0(\rho,z)}(d\rho^2 + dz^2) + \rho^2 d\phi^2 \right] \tag{8}$$

there exists a one-parameter generalization:

$$\begin{aligned} ds^2 &= \pm e^{2u_0(\rho,z)} \rho^{2\alpha} dt^2 + e^{-2(1-2\alpha)u_0(\rho,z)} \rho^{2\alpha(\alpha-1)} \left[e^{2k_0(\rho,z)}(d\rho^2 + dz^2) \right] \\ &+ e^{-2u_0(\rho,z)} \rho^{2(1-\alpha)} d\phi^2. \end{aligned} \tag{9}$$

Group Properties and Solution Space

- Denoting our transformation by T_α, one can check $T_{\alpha_2} \circ T_{\alpha_1} = T_{\alpha_1+\alpha_2}$ (closure). The seed metric is the identity $\alpha = 0$ (any metric within the family can be taken to be at $\alpha = 0$). $[T_\alpha]^{-1} = T_{-\alpha}$. Thus our transformation is a Lie point symmetry.
- Existence of Lie point symmetries is expected given the linearity (2) and that it determines the whole system: for a given u_1 the superposition $u = u_0 + \alpha u_1$ maps the whole solution space into itself. The linearity also means α can take any value, so the resulting (u, k) from a specific (u_0, k_0) would represent an infinite family of solutions – a curve in the space of solutions parametrized by α.
- What Ernst noticed is that superposing $u_1 = z$ to u_0 creates additive terms for k_0 that can be obtained via the simpler equation.
- What we found is that if one instead takes $u_1 = \ln \rho$, one obtains an *explicit* algebraic prescription for k without having to solve any associated set of equations.

Warped Form

- Many physically and mathematically interesting solutions with two commuting hypersurface-orthogonal Killing vector fields come in different coordinates and/or signatures, and may not possess axial symmetry. Our symmetry (7) can be rewritten as transforming the general warped product Ricci-flat solution
$$ds^2 = \pm g_{11}(z^i)dx^2 \pm g_{22}(z^i)dy^2 + g_{ij}(z^i)dz^i dz^j, \quad i,j = 3,4, \tag{10}$$
with two line fibres corresponding to the two KVs $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, to
$$\begin{aligned} ds^2 = \;& \pm \;\; (g_{22})^{\gamma}(g_{11})^{\gamma} g_{11} dx^2 \pm (g_{22})^{-\gamma}(g_{11})^{-\gamma} g_{22} dy^2 \\ & + \;\; (g_{22})^{\gamma(\gamma-1)}(g_{11})^{\gamma(\gamma+1)} g_{ij} dz^i dz^j, \end{aligned} \tag{11}$$
which is Ricci-flat for $\gamma \in (-\infty, \infty)$.
- This is just a rewrite of the above Theorem with ρ written as $\sqrt{g_{11}g_{22}}$ and $\alpha = \gamma$ (and other coordinates accordingly identified). The slightly elaborate form of the metric components in is deliberate, to make the exponent structure manifest.

Example of New Solutions

- Applying T_γ to the Schwarzschild metric we obtain

$$\begin{aligned} ds^2 &= -r^{2\gamma}(\sin\theta)^{2\gamma}\left(1-\frac{2m}{r}\right)^{\gamma+1}dt^2 \\ &+ r^{2\gamma^2-2\gamma}(\sin\theta)^{2\gamma^2-2\gamma}\left(1-\frac{2m}{r}\right)^{\gamma^2+\gamma-1}dr^2 \\ &+ r^{2\gamma^2-2\gamma+2}(\sin\theta)^{2\gamma^2-2\gamma}\left(1-\frac{2m}{r}\right)^{\gamma^2+\gamma}d\theta^2 \\ &+ r^{2-2\gamma}(\sin\theta)^{2-2\gamma}\left(1-\frac{2m}{r}\right)^{-\gamma}d\phi^2. \end{aligned} \tag{12}$$

- Generalization of the C-metric is also direct. Applying T_α to $(u_0, k_0) \equiv (0, 0)$, i.e. to Minkowski in polar coordinates, we get the Levi-Civita metric $ds^2 = -\rho^{2\alpha}dt^2 + \rho^{2\alpha^2-2\alpha}(d\rho^2 + dz^2) + \rho^{-2\alpha+2}d\phi^2$.
- There are plenty of other solutions, including cylindrical gravitational wave solutions, on which this can be applied equally easily.

Axially-Symmetric Static Metrics in vacuum

Historical Notes:

- Hans Buchdahl (1954) showed that if

$$ds^2 = g_{ik}(x^j)dx^i dx^k + g_{aa}(x^j)(dx^a)^2, \qquad (13)$$

which is "static" in x^a, is Ricci-flat then so is its "reciprocal transform"

$$ds_r^2 = (g_{aa})^{2/(n-3)}(x^j)g_{ik}dx^i dx^k + (g_{aa})^{-1}(x^j)(dx^a)^2. \qquad (14)$$

- Also noted that if there are more "static coordinates" more general solutions "can be formed by means of a succession of reciprocal transformations, starting with the line element of a flat space" which he actually applied in a much later paper in 1978.
- However writing flat space in various coordinates all solutions Buchdahl obtained, were known already.

Axially-Symmetric Static Metrics in vacuum

- Applying the transformation to the Schwarzschild metric Buchdahl obtained

$$ds^2 = -\frac{dt^2}{1-\frac{2m}{r}} + \left(1-\frac{2m}{r}\right) dr^2 + r^2 \left(1-\frac{2m}{r}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

  which, with $R = r - 2m$, is the Schwarzschild metric with mass $-m$.
- Buchdahl missed the possibility of using ϕ-coordinate. Also alternating between t and ϕ as he did in his 1978 paper to obtain (known) solutions from the flat metric, it is conceivable that Buchdahl could have arrived at our generalized Schwarzschild metric more than 30 years ago. More importantly, he did not look for an explanation of why for two static coordinates the discrete exponents produced by alternate transformations also work fine for continuous values.
- It is easy to see that $\alpha = \gamma = 1$ in our transformation gives Buchdahl's reciprocal transformation. So does $\beta = -1$ of the scaling symmetry (6) in Weyl coordinates.

Generalizing Ricci Solitons

The symmetry of the statics system enables us to construct new steady Ricci solitons.

- A Ricci soliton is a manifold-with-metric $(M^{n+1}, g_{\mu\nu})$ and vector field X on it such that

$$R_{\mu\nu} - \frac{1}{2}\mathcal{L}_X g_{\mu\nu} = \kappa g_{\mu\nu} \tag{15}$$

for some constant κ. The soliton is "steady" if $\kappa = 0$, "expander" if $\kappa < 0$, "shrinker" if $\kappa > 0$.

- They arise as self-similar solutions of the Ricci flow equations

$$\frac{\partial g_{\mu\nu}}{\partial \eta} = -2R_{\mu\nu}, \tag{16}$$

in which the metric evolves only by rescalings and diffeomorphisms $g_{\mu\nu}(\eta) = \sigma(\eta)\psi^*_\eta(g_{\mu\nu}(0))$ from which (15) follows.

- A soliton is called gradient if $X = \nabla f$, thus (15) becomes

$$R_{\mu\nu} - \nabla_\mu \nabla_\nu f = \kappa g_{\mu\nu}. \tag{17}$$

An Example of Ricci Soliton

- The Cigar soliton, or Witten's black hole, is a simple but nontrivial Ricci soliton, where
$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}, \tag{18}$$
and $X = 2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$. It is a steady soliton on $\mathbb{R}^2$ solving (15) with $\kappa = 0$ and is gradient with $f = x^2 + y^2$.
- It can be written in various other coordinates:
$$\begin{aligned} ds^2 &= \frac{dr^2 + r^2 d\theta^2}{1 + r^2} = ds^2 + \tanh^2(s) d\theta^2 && (19) \\ &= \left(1 - \frac{M}{\rho}\right) d\theta^2 + \left(1 - \frac{M}{\rho}\right)^{-1} \frac{d\rho^2}{\rho^4} && (20) \\ &= \frac{dz^2 + d\theta^2}{1 + e^{-2z}} && (21) \end{aligned}$$
- All nontrivial steady gradient solitons are non-compact.

Ricci Solitons and Static Metrics

- It is well-known that if

$$ds^2 = \pm e^{2u} dt^2 + e^{-\frac{2u}{n-2}} g_{ij} dx^i dx^j \tag{22}$$

is Ricci-flat in $(n+1)$-dimensions then (u, g_{ij}) solves the Einstein scalar field equations in n-dimensions

$$R_{ij} - \frac{n-1}{n-2} \nabla_i u \nabla_j u = 0, \ \Delta u = 0.$$

- (Akbar, Woolgar, CQG **26**, 2009) Every solution of Einstein-scalar field theory with a possible cosmological constant corresponds to a Ricci soliton in one higher dimensions. This for vanishing cosmological constant means every $(n+1)$-dimensional static vacuum solution (22) can be put in one-to-one correspondence with the following Ricci soliton metric in $(n+1)$-dimensions with $X := -2\sqrt{\frac{n-1}{n-2}} g^{ij} \nabla_i u \frac{\partial}{\partial x^j}$:

$$ds^2 = e^{2\sqrt{\frac{n-1}{n-2}} u} dt^2 + g_{ij} dx^i dx^j. \tag{23}$$

Ricci Solitons and Static Metrics

- Steady solitons generated this way are necessarily incomplete in four dimensions (follows from the inability of the Einstein-scalar system to admit any complete non-flat solution (Anderson 1999)).
- Corresponding to every axisymmetric vacuum solution of the Einstein equations in Weyl coordinates (1) we therefore have the following local Ricci soliton

$$ds^2 = \pm e^{2\sqrt{2}u} dt^2 + \left[e^{2k(\rho,z)} (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right] \tag{24}$$

with $X = -2\sqrt{2} e^{-2k(\rho,z)} \left(\nabla_\rho u \frac{\partial}{\partial \rho} + \nabla_z u \frac{\partial}{\partial z} \right)$.

One-Parameter Generalization

- Finally, using the correspondence above and our Lie point symmetry (7) we obtain the following one-parameter family of local steady Ricci solitons

$$ds^2 = \pm e^{2\sqrt{2}u}\rho^{2\sqrt{2}\alpha}dt^2 + \left[e^{2k(\rho,z)+4\alpha u(\rho,z)}\rho^{2\alpha^2}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right]$$

with $X = -2\sqrt{2}e^{-2k(\rho,z)+2\alpha u(\rho,z)+\alpha^2 \ln \rho}\left(\frac{\alpha}{\rho}\frac{\partial}{\partial\rho} + \nabla_\rho u \frac{\partial}{\partial\rho} + \nabla_z u \frac{\partial}{\partial z}\right)$ for every static axisymmetric vacuum solution of the Einstein equations (1).

Conclusion and Future Directions

- We found a nontrivial exact Lie point symmetry in explicit algebraic form. Being a symmetry of the system it applies equally to exact and non-exact solutions of the system. The symmetry generalizes the corresponding Ricci solitons simultaneously.
- One can generalize any axisymmetric static or cylindrical gravitational wave solution and there is a plethora of possibilities. One can obtain multi-parameter solutions by combining it with the scaling symmetry, and with Ernst's prescription and study their properties, interpret.
- However, the more important message that we believe comes from this unexpected result is that looking vigorously and systematically for further hidden symmetries of the static system, and obtaining a clearer picture of the geometry of the solution space, would be worthwhile. This falls within the purview of symmetry analysis of nonlinear PDEs, a very developed field in mathematics, and would possibly connect with the stationary system. This is work in progress.

References

- M. M. Akbar and E. Woolgar, "Ricci Solitons and Einstein-Scalar Field Theory," Class. Quant. Grav. **26**, 055015 (2009)
- M T Anderson, "Scalar Curvature, Metric Degenerations and the Static Vacuum Einstein Equations on 3-manifolds, I", Geom Funct Anal 9 (1999) 855.
- H. A. Buchdahl, "Reciprocal Static Solutions of the Eqauations $G_{\mu\nu}$," Quart. J. Math. Oxford Ser 2 **5** (1954), 116-19.
- H. A. Buchdahl, "Reciprocal Static Solutions of the Equations of the Gravitational Field," Austral. J. Phys. **9** (1956), 1318.
- H. A. Buchdahl, "Reciprocal Static Metrics and Scalar Fields in the General Theory of Relativity," Phys. Rev. **115** (1959) 1325.
- H. A. Buchdahl, "On Solutions of Einstein's Equations with Scalar Zero-Mass Source," Gen. Rel. Grav. **9**, 59 (1978).

References

B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, Mathematical Surveys and Monographs vol 110 (AMS, Providence, 2004).

B. Chow et al, *The Ricci Flow: Techniques and Applications: Geometric Aspects*, Mathematical Surveys and Monographs vol 135 (AMS, Providence, 2007).

J. Ehlers, Konstruktionen und Charakterisierungen von Lösungen der Einsteinschen Gravitationsfeldgleichungen, Dissertation, Hamburg, 1957.

V. Ernst, "Generalized C-metric," J. Math. Phys. **19**, 489 (1978).

F. J. Ernst, "Erratum: Generalized C Metric [J. Math. Phys. 1 9, 1986(1978)]," J. Math. Phys. **20**, 2371 (1979).

R. P. Geroch, "A Method for Generating Solutions of Einstein's Equations," J. Math. Phys. **12**, 918 (1971).

References

H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt (2003) *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, UK).

H. Stephani, "Differential Equations: Their Solutions Using Symmetries," Cambridge, UK: Univ. Pr. (1989) 260 p

P. J. Olver (1993). *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New York).

A. Tomimatsu and H. Sato, "New Exact Solution for the Gravitational Field of a Spinning Mass," Phys. Rev. Lett. **29**, 1344 (1972).

A. Tomimatsu and H. Sato, "Multi-Soliton Solutions of the Einstein Equation and the Tomimatsu-Sato Metric," Prog. Theor. Phys. Suppl. **70**, 215 (1981).