Resistance Distance in Discretized Gravity

Mikhail Kagan
Penn State Abington

Outline

- **Motivation**

- **Resistance distance in graph theory**

- **Generalized resistance distance**

- **Random Walks on Graphs**

- **Possible applications in discretized gravity**

Motivation

- **Chemistry of complex molecules**

- **Electric circuits of various topology**

- **Random walks**

- **Another type of metric on graphs ->**
 **applications to gravity?**

# Resistance Distance. Examples

$d_{ab} = 2$

$d_{ab} = 2$
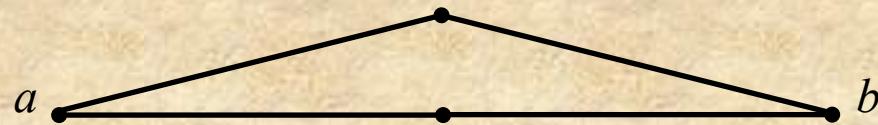
$d_{ab} = 2$
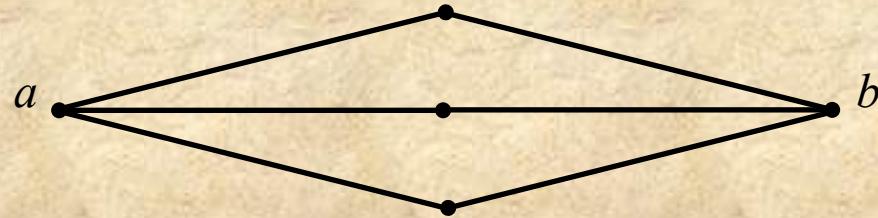
Resistance Distance. Examples
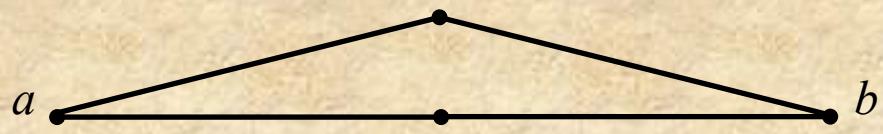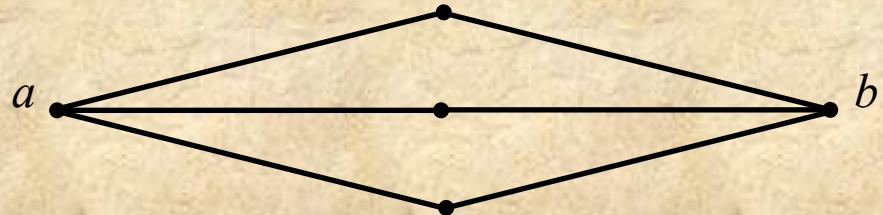
$d_{ab} = 2$ a •———————•———————• b $R_{ab} = 2$

$d_{ab} = 2$

$d_{ab} = 2$

Resistance Distance. Examples

$d_{ab} = 2$ a •———————•———————• b $R_{ab} = 2$

$d_{ab} = 2$ a b $R_{ab} = 1$

$d_{ab} = 2$ a b

Resistance Distance. Examples

$d_{ab} = 2$ a •————————•————————• b $R_{ab} = 2$

$d_{ab} = 2$ a ◢——————◣ b $R_{ab} = 1$
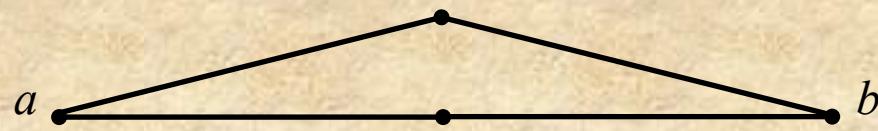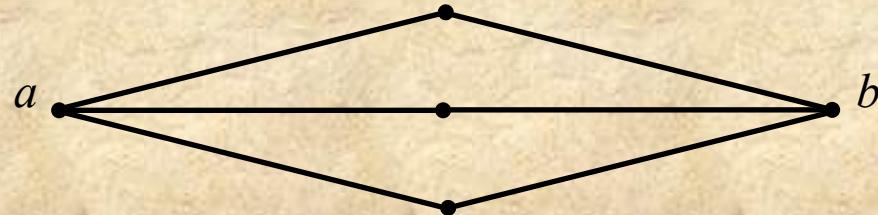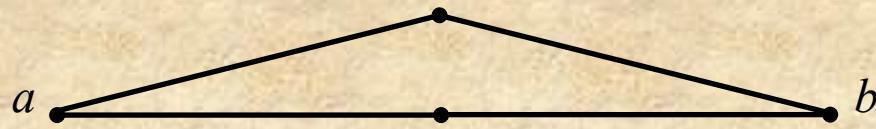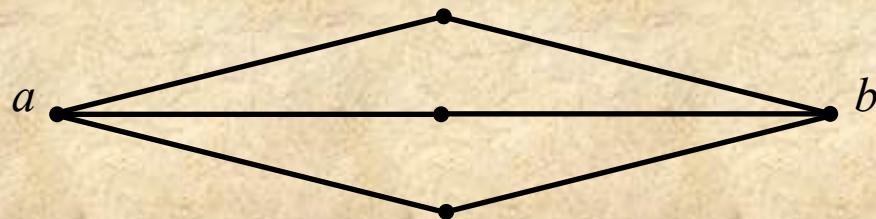
$d_{ab} = 2$ a ◆ b $R_{ab} = 2/3$

Resistance Distance. Metric

Resistance distance satisfies (Klein 1993)

$$R\,(a, b) \geq 0$$

$$R\,(a, b) = 0, \quad \text{iff} \quad a=b$$

$$R\,(a, b) = R\,(b, a)$$

$$R\,(a, b) \leq R\,(a, c) + R\,(c, b)$$

Resistance Distance. Metric

Resistance distance satisfies (Klein 1993)

$$R\,(a,\,b) \geq 0$$

$$R\,(a,\,b) = 0, \quad \text{iff} \quad a{=}b$$

$$R\,(a,\,b) = R\,(b,\,a)$$

$$R\,(a,\,b) \leq R\,(a,\,c) + R\,(c,\,b)$$

Standard Description

Adjacency matrix: $\mathbf{A}_{ij} = \mathbf{A}_{ji} = \mathbf{1}$ or $\mathbf{0}$ (if connected or not)

Degree matrix: $\mathbf{D}_{ij} = diag\,(d_1,\,d_2,\,\ldots),\,d_i = \Sigma_j \mathbf{A}_{ij}$

Laplacian matrix: $\mathbf{L} = \mathbf{D} - \mathbf{A}$

Resistance Distance. Metric

Resistance distance satisfies (Klein 1993)

$$R\,(a,\,b) \geq 0$$

$$R\,(a,\,b) = 0, \quad \text{iff} \quad a=b$$

$$R\,(a,\,b) = R\,(b,\,a)$$

$$R\,(a,\,b) \leq R\,(a,\,c) + R\,(c,\,b)$$

Standard Description

Adjacency matrix: $\mathbf{A}_{ij} = \mathbf{A}_{ji} = \mathbf{1}$ or $\mathbf{0}$ (if connected or not)

Degree matrix: $\mathbf{D}_{ij} = diag\,(d_1, d_2, \ldots), d_i = \Sigma_j \mathbf{A}_{ij}$

Laplacian matrix: $\mathbf{L} = \mathbf{D} - \mathbf{A}$

Resistance distance

$$R_{ab} = \Gamma_{aa} + \Gamma_{bb} - 2\Gamma_{ab}$$

Γ = Moore-Penrose pseudo inverse Laplacian matrix

Resistance Distance. Generic graph

Edge conductance $\quad \sigma_{ij} = \dfrac{1}{R_{ij}} \geq 0 \qquad c_i := \displaystyle\sum_{j=1}^{n} \sigma_{ij}$

Resistance Distance. Generic graph

Edge conductance $\qquad \sigma_{ij} = \dfrac{1}{R_{ij}} \geq 0 \qquad c_i := \displaystyle\sum_{j=1}^{n} \sigma_{ij}$

Generalized (weighted) Laplacian

$$
\begin{pmatrix}
c_1 & -\sigma_{12} & -\sigma_{13} & \cdots & -\sigma_{1,n-1} & -\sigma_{1,n} \\
-\sigma_{21} & c_2 & -\sigma_{23} & \cdots & -\sigma_{2,n-1} & -\sigma_{2,n} \\
-\sigma_{31} & -\sigma_{32} & c_3 & \cdots & -\sigma_{3,n-1} & -\sigma_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\sigma_{n-1,1} & -\sigma_{n-1,2} & -\sigma_{n-1,3} & \cdots & c_{n-1} & -\sigma_{n-1,n} \\
-\sigma_{n,1} & -\sigma_{n,2} & -\sigma_{n,3} & \cdots & -\sigma_{n,n-1} & c_n
\end{pmatrix}
$$

Resistance Distance. Generic graph

Edge conductance $\qquad \sigma_{ij} = \dfrac{1}{R_{ij}} \geq 0 \qquad c_i := \displaystyle\sum_{j=1}^{n} \sigma_{ij}$

Generalized (weighted) Laplacian $\qquad \overset{\displaystyle\sum\nolimits'}{\curvearrowleft}$

$$\begin{pmatrix} c_1 & -\sigma_{12} & -\sigma_{13} & \cdots & -\sigma_{1,n-1} & -\sigma_{1,n} \\ -\sigma_{21} & c_2 & -\sigma_{23} & \cdots & -\sigma_{2,n-1} & -\sigma_{2,n} \\ -\sigma_{31} & -\sigma_{32} & c_3 & \cdots & -\sigma_{3,n-1} & -\sigma_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\sigma_{n-1,1} & -\sigma_{n-1,2} & -\sigma_{n-1,3} & \cdots & c_{n-1} & -\sigma_{n-1,n} \\ -\sigma_{n,1} & -\sigma_{n,2} & -\sigma_{n,3} & \cdots & -\sigma_{n,n-1} & c_n \end{pmatrix}$$

Resistance Distance. Generic graph

Edge conductance
$$\sigma_{ij} = \frac{1}{R_{ij}} \geq 0 \qquad c_i := \sum_{j=1}^{n} \sigma_{ij}$$

Generalized (weighted) Laplacian Σ'

$$\begin{pmatrix} c_1 & -\sigma_{12} & -\sigma_{13} & \cdots & -\sigma_{1,n-1} & -\sigma_{1,n} \\ -\sigma_{21} & c_2 & -\sigma_{23} & \cdots & -\sigma_{2,n-1} & -\sigma_{2,n} \\ -\sigma_{31} & -\sigma_{32} & c_3 & \cdots & -\sigma_{3,n-1} & -\sigma_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\sigma_{n-1,1} & -\sigma_{n-1,2} & -\sigma_{n-1,3} & \cdots & c_{n-1} & -\sigma_{n-1,n} \\ -\sigma_{n,1} & -\sigma_{n,2} & -\sigma_{n,3} & \cdots & -\sigma_{n,n-1} & c_n \end{pmatrix}$$

Σ''

Resistance Distance. Generic graph

Edge conductance $\qquad \sigma_{ij} = \dfrac{1}{R_{ij}} \geq 0 \qquad c_i := \displaystyle\sum_{j=1}^{n} \sigma_{ij}$

Generalized (weighted) Laplacian $\qquad \Sigma'$

$$\begin{pmatrix} c_1 & -\sigma_{12} & -\sigma_{13} & \cdots & -\sigma_{1,n-1} & -\sigma_{1,n} \\ -\sigma_{21} & c_2 & -\sigma_{23} & \cdots & -\sigma_{2,n-1} & -\sigma_{2,n} \\ -\sigma_{31} & -\sigma_{32} & c_3 & \cdots & -\sigma_{3,n-1} & -\sigma_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\sigma_{n-1,1} & -\sigma_{n-1,2} & -\sigma_{n-1,3} & \cdots & c_{n-1} & -\sigma_{n-1,n} \\ -\sigma_{n,1} & -\sigma_{n,2} & -\sigma_{n,3} & \cdots & -\sigma_{n,n-1} & c_n \end{pmatrix}$$

Σ''

Equivalent conductance (MK 2015)

$$\sigma_{\mathrm{eq}} = \frac{\det \Sigma'}{\det \Sigma''}$$

15

Resistance Distance. Generic graph

Properties

- σ_{eq} has the right permutation symmetry

- For a connected circuit   det Σ' > 0,   det Σ'' > 0
  connectivity characterized by det Σ' and det Σ''

- det Σ' or det Σ'' (σ_{ij}) = $A\sigma_{ij} + B$
  (linear in every edge conductance)

- If σ_{eq} is the same for σ_{ij} = 0 or ∞ then it doesn't depend on σ_{ij}

- Z_{eq} in AC-circuits can model +/- resistance

Resistance Distance. Generic graph

For arbitrary two vertices (a, b)

det Σ' does not change

det Σ'' obtained from Σ by crossing out (a,b)

$$\det \Sigma''_{ab} = \frac{\partial}{\partial \sigma_{ab}} \det \Sigma'$$

Resistance Distance. Generic graph

For arbitrary two vertices (a, b)

det Σ' does not change

det Σ'' obtained from Σ by crossing out (a,b)

$$\det \Sigma''_{ab} = \frac{\partial}{\partial \sigma_{ab}} \det \Sigma'$$

Equivalent resistance/Resistance distance

$$R_{ab} = \frac{\partial}{\partial \sigma_{ab}} \ln \det \Sigma'$$

Random Walks on Graphs

Transition probability matrix

$$P_{ij} = \frac{\sigma_{ij}}{c_i}$$

Random Walks on Graphs

Transition probability matrix

$$P_{ij} = \frac{\sigma_{ij}}{c_i}$$

By construction

$$P_{ij} \geq 0, \quad \sum_j P_{ij} = 1$$

Random Walks on Graphs

Transition probability matrix

$$P_{ij} = \frac{\sigma_{ij}}{c_i}$$

By construction

$$P_{ij} \geq 0, \quad \sum_j P_{ij} = 1$$

Escape probability

$$P_{ab}^{\text{esc}} = \frac{\sigma_{ab}^{\text{eq}}}{c_a}$$

Possible applications in discretized gravity

- **Regge calculus**

- **Quantum graphity**

- **Modeling space-time with complex impedance**

    Issues:

    i) **parallel resonance -> diverging impedance distance**

    ii) **Polya's theorem about 3D (and higher) lattices**