Singularity Theorems in Regularity $C^{1,1}$

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joint work with

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Blueprint of the generic singularity theorem (J. Senovilla):

Th. If g is C^2 then (i)–(iv) are incompatible, where

(i) Energy condition.

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- Below $C^{1,1}$: unbounded curvature, non-unique geodesics: singular.
- Hence $C^{1,1}$ is the natural threshold for singularity theorems.

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Penrose, 1965

Let (M, g) be a C^2- spacetime such that

(i) $\mathrm{Ric}(X, X) \geq 0$ for every null vector X.

(ii) There exists a non-compact Cauchy hypersurface S in M

(iii) There exists a compact achronal spacelike submanifold of codimension 2 with past-pointing timelike mean curvature vector field.

Then M is future null geodesically incomplete.

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- Employ regularization adapted to causal structure following ▸ Chrusciel/Grant: $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$, $\check{g}_\varepsilon, \hat{g}_\varepsilon \to g$ in C^1, curvatures loc. unif. bounded.

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- Avoid Calculus of Variations.
- Re-build causality theory for $\mathcal{C}^{1,1}$-metrics.

The $\mathcal{C}^{1,1}$-proof

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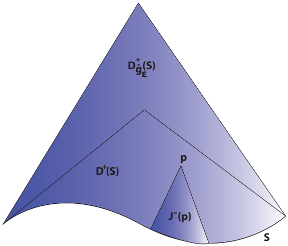
- $D^+(S) \subseteq D^+_{\check{g}_\varepsilon}(S)$:



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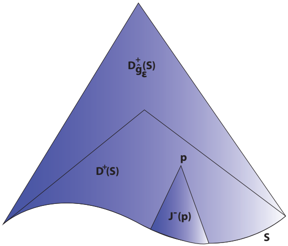

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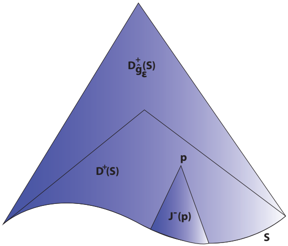

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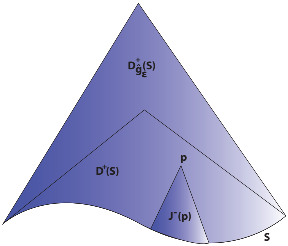

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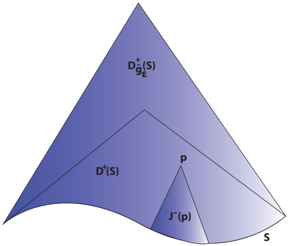

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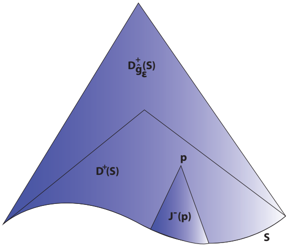

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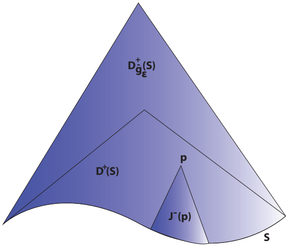

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- Derive a contradiction as in the $\mathcal{C}^\infty$-case.

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Related results, comparison geometry

- Riemannian comparison results: Ricci, mean curvature, Laplace, ...
- Grant (2011): Null-cone comparison theorems
- Grant/Treude (2013): Lorentzian volume comparison with model spaces (warped products), new proof of Hawking's theorem
- Graf (2016): Comparison geometry proof of $C^{1,1}$-Hawking theorem.
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Current research

- Comparison approach to Penrose singularity theorem (Evolve trapped surface along null geodesics, quantify area), should also give new proof in $C^{1,1}$.
- Long term goal: Hawking-Penrose singularity theorem in $C^{1,1}$: Will require completely new methods.

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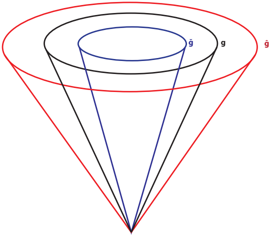
Chrusciel-Grant regularization of the metric



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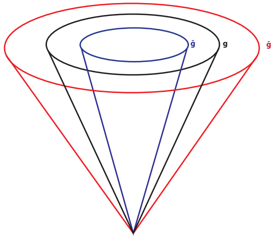

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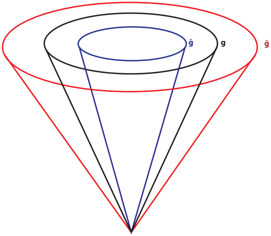

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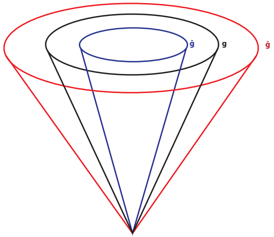

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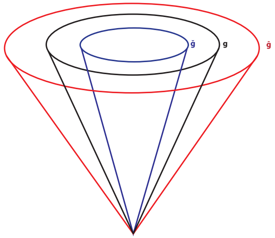

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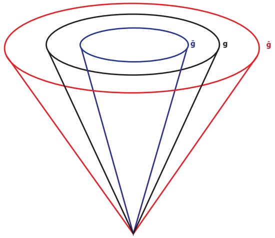

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- Glue w.r.t. x and ε to obtain Lorentzian metric $\check{g}_\varepsilon \prec g$. Back

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$$\forall C > 0 \ \forall \delta > 0 \ \forall \kappa < 0 \ \forall^0 \varepsilon \ \forall X \in TM|_K \text{ with } \check{g}_\varepsilon(X,X) \leq \kappa, \ \|X\|_h \leq C : \\ \mathrm{Ric}_\varepsilon(X,X) > -\delta.$$

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Proof.

- $\check{g}_\varepsilon - g * \rho_\varepsilon \to 0$ in $\mathcal{C}^2 \rightsquigarrow$ suffices to consider $g_\varepsilon := g * \rho_\varepsilon$.
- $R_{jk} = R^i_{jki} = \partial_{x^i}\Gamma^i_{kj} - \partial_{x^k}\Gamma^i_{ij} + \Gamma^i_{im}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{ij}$
- Blue terms$|_\varepsilon$ converge uniformly.
- For red terms use variant of Friedrich's Lemma:

$$(R_{jk}X^jX^k) * \rho_\varepsilon - R^\varepsilon_{jk}X^jX^k \to 0 \text{ uniformly}$$

$$\rho_\varepsilon \geq 0 \Rightarrow (R_{jk}X^jX^k) * \rho_\varepsilon \geq 0.$$

$\square$ Back

The Hawking-Penrose Singularity Theorem

Hawking and Penrose, 1967

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1. Every inextendible causal geodesic contains a pair of conjugate points,
2. M contains no closed timelike curves and
3. there is a future or past trapped achronal set S.

Corollary

M must be causally incomplete if Einstein's equations hold and

1. M contains no closed timelike curves.
2. M satisfies an energy condition.
3. *Generality*: nontrivial curvature at some point of any causal geodesic.
4. M contains either
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