A geometric characterisation of rotating black hole spacetimes using Killing spinors

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Background

- The Kerr-Newman solution - important mathematically and physically
- Characterisations of black hole spacetimes can be used to
    - understand black hole uniqueness theorems
    - study evolution of perturbations
    - illustrate fundamental properties of black hole solutions
- We focus on a characterisation of Kerr-Newman using Killing spinors, and use it to construct a geometric invariant for an initial data set measuring the deviation from exact Kerr-Newman data

Killing spinors

- A Killing spinor is a valence-2 symmetric spinor κ_{AB} satisfying

$$\nabla_{A'(A}\kappa_{BC)} = 0$$

- Existence of a Killing spinor gives strong restrictions on the algebraic (Petrov) type of the spacetime:

$$\psi_{ABCD} \propto \kappa_{(AB}\kappa_{CD)}$$

- Related to Killing Yano tensors, representing 'hidden symmetries' of the spacetime (e.g. Carter constant for Kerr)

- $\xi_{AA'} := \nabla^{B}{}_{A'}\kappa_{AB}$ is a Killing vector in 'matter aligned' spacetimes (e.g. vacuum)

Theorem (MC, Valiente Kroon)

Let $(\mathcal{M}, g, F)$ be a smooth, electrovacuum spacetime satisfying the matter alignment condition, with a stationary asymptotically flat end $\mathcal{M}_\infty$ generated by a Killing spinor κ_{AB}. Let both the Komar mass associated to the Killing vector $\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}$ and the total electromagnetic charge in $\mathcal{M}_\infty$ be non-zero. Then $(\mathcal{M}, g, F)$ is locally isometric to the Kerr-Newman spacetime.

3+1 decomposition

- We now perform the well-known 3+1 decomposition:
  - The spacetime is foliated by a family of spacelike hypersurfaces Σ_t, $t \in \mathbb{R}$
  - The induced metric h_{ab} and extrinsic curvature K_{ab} satisfy the constraint equations on Σ_t
- We wish to encode the existence of a Killing spinor as conditions intrinsic to the hypersurface Σ_0.
- Example: Integrability condition for Killing spinor equation:

$$\Psi_{F(ABC}\kappa_{D)}{}^{F} = 0$$

  Algebraic condition, must be satisfied on Σ_0.
- Vacuum case done by Bäckdahl & Kroon, electrovacuum case follows a similar procedure.

Integrability condition for the Killing spinor

Note that:

$$\nabla_{A'(A}\kappa_{BC)} = 0 \quad \Rightarrow \quad \Box\kappa_{AB} + \psi_{ABCD}\kappa^{CD} = 0 \tag{1}$$

We make use of the following result:

Theorem

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime, with a Cauchy surface Σ. Then on the the spacetime $(\mathcal{M}, g)$, there exists a unique solution $\{T_i, i = 1, ..., n\}$ to the problem:

$$\Box T_i = f_i\left(T_1, ..., T_n\right)$$
$$T_i|_\Sigma = g_i, \quad \nabla T_i|_\Sigma = h_i$$

for $i = 1, 2, ..., n$, where f_i is homogeneous in each T_i and their first derivatives, and g_i, h_i are freely specifiable functions on Σ.

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Finding a Killing spinor

Assume κ_{AB} satisfies the evolution condition, and define:

$$H_{A'ABC} = \nabla_{A'(A}\kappa_{BC)}$$
$$S_{AA'BB'} = \nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'}$$
$$\Theta_{AB} = \kappa_{(A}{}^{C}\phi_{B)C}$$
$$\zeta_{AA'} = \nabla^{B}{}_{A'}\Theta_{AB}$$

Here ϕ_{AB} is the Maxwell spinor, the spinorial version of the Faraday tensor. Then,

$$\Box H_{AA'BB'} = \alpha\left(H, S, \Theta, \zeta\right)$$
$$\Box S_{AA'BB'} = \beta\left(H, S, \Theta, \zeta\right)$$
$$\Box \Theta_{AB} = \gamma\left(H, S, \Theta, \zeta\right)$$
$$\Box \zeta_{AA'} = \delta\left(H, S, \Theta, \zeta\right)$$

where $\alpha, \beta, \gamma, \delta$ are homogeneous in H, S, Θ, ζ and their first derivatives.

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Finding a Killing spinor

Lemma

The spacetime obtained from an electrovacuum initial data set (Σ_0, h, K, ϕ) admits a Killing spinor if on Σ_0 we have:

$$H_{A'ABC} = \nabla H_{A'ABC} = S_{AA'BB'} = \nabla S_{AA'BB'} = 0$$
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These initial conditions for H, S, Θ, ζ can be decomposed into purely spatial spinorial objects on the hypersurface Σ.

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The spacetime obtained from an electrovacuum initial data set (Σ_0, h, K, ϕ) admits a Killing spinor if on Σ_0 we have:

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0$$

$$\psi_{(ABC}{}^{F}\kappa_{D)F} = 0$$

$$\kappa_{(A}{}^{C}\phi_{B)C} = 0$$

$$\mathfrak{Q}^2 \equiv \left(\kappa_{AB}\kappa^{AB}\right)^2 \phi_{AB}\phi^{AB} = \text{constant}$$

The Killing spinor is obtained by evolving the initial data $(\kappa_{AB}, \nabla\kappa_{AB})$ for the Killing spinor satisfying the above equations and

$$\nabla\kappa_{AB} = -\mathcal{D}_{(A}{}^{C}\kappa_{B)C}$$

using the evolution equation:

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- The key equation in this characterisation is the *spatial Killing spinor equation*:

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0.$$

This equation is overdetermined, so does not in general admit a solution for an arbitrary initial data set.

- Key idea: define map $\Phi : \kappa_{AB} \mapsto \mathcal{D}_{(AB}\kappa_{CD)}$, and construct adjoint Φ^* with respect to inner product given by:

$$\langle \lambda, \nu \rangle \equiv \int_{\Sigma_0} \lambda_{AB}\hat{\nu}^{AB} \mathrm{d}\mu$$

Then, the operator

$$L \equiv \Phi^* \circ \Phi : \kappa_{AB} \mapsto \mathcal{D}^{CD}\mathcal{D}_{(CD}\kappa_{AB)} - \Omega^{CDF}{}_{(A}\mathcal{D}_{B)F}\kappa_{CD} - \Omega^{CDF}{}_{(A|}\mathcal{D}_{CD}\kappa_{|B)F}$$

is a self adjoint elliptic operator. The equation

$$L(\kappa_{AB}) = 0$$

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The geometric invariant

- We define the geometric invariant by:

$$\mathcal{I} = ||\mathcal{D}_{(AB}\kappa_{CD)}||^2 + ||\psi_{(ABC}{}^{F}\kappa_{D)F}||^2 + ||\kappa_{(A}{}^{C}\phi_{B)C}||^2 + ||\mathcal{D}_{AB}\mathfrak{Q}^2||^2$$

where the norm $||\cdot||^2 \equiv \langle \cdot , \cdot \rangle$.

Theorem (MC, Valiente Kroon)

Let (Σ_0, h, K, ϕ) be a smooth, asymptotically flat initial data set for the Einstein-Maxwell equations such that:

- There exist coordinates in the asymptotic end such h, K are asymptotically Schwarzschildean
- The Komar mass and charge are non-zero.

Let κ_{AB} be the solution to $L(\kappa_{AB}) = 0$ on Σ_0. Then the invariant $\mathcal{I}$ defined above vanishes iff (Σ_0, h, K, ϕ) is locally an initial data set for the Kerr-Newman spacetime.