Geometric Inequalities Involving Mass, Angular Momentum, and Charge

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Classical Penrose Inequality with Angular Momentum

It is believed that the general picture for the long time evolution of spacetime consists of two main parts. Whenever singularities develop (which is a generic phenomenon), they must always be hidden inside black holes. Moreover, spacetime will eventually settle down to a collection of stationary rotating black holes, which in electrovacuum must be given by Kerr-Newman solutions.

Classical Penrose Inequality with Angular Momentum

It is believed that the general picture for the long time evolution of spacetime consists of two main parts. Whenever singularities develop (which is a generic phenomenon), they must always be hidden inside black holes. Moreover, spacetime will eventually settle down to a collection of stationary rotating black holes, which in electrovacuum must be given by Kerr-Newman solutions. Since gravitational radiation carries away positive energy, the mass of any initial state should be greater than or equal to the mass of the final state $m_i \geq m_f$. Moreover, the Hawking area theorem ensures that the area of horizons is nondecreasing $A_i \leq A_f$. Thus, if conditions are imposed to ensure conservation of angular momentum (such as axisymmetry and electrovacuum) and charge (absence of charged matter), then a basic inequality relating these quantities in any initial data set may be (heuristically) derived from the relation they satisfy within the Kerr-Newman spacetime.

$$
\begin{aligned}
m_i^2 \geq m_f^2 & \\
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The last inequality follows because, as a function of A, the expression on the right-hand side is nondecreasing precisely when the auxiliary area-angular momentum-charge inequality is satisfied. For single stable apparent horizons, the auxiliary inequality is known to hold, and thus may be ignored in this situation. By setting $q_i = 0$ we then obtain the desired inequality

$$
m_i^2 \geq \frac{A_i}{16\pi} + \frac{4\pi\mathcal{J}_i^2}{A_i}.
$$

Consider an axisymmetric initial data set (M^3, g, k) with one end that is asymptotically flat, and with a connected minimal surface boundary. If M^3 is simply connected, Chrusciel and Nguyen have shown that $M^3 \setminus \partial M^3 \cong \mathbb{R}^3 \setminus I$ and there exists a global 'Weyl' coordinate system such that

$$g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\phi + A_\rho d\rho + A_z dz)^2,$$

where $\eta = \partial_\phi$ is the Killing field.
These Weyl coordinates give the same metric structure as Brill coordinates, but only cover the region outside the horizon. Here the orbit space consists of the upper half plane $\mathbb{R}^2_+ = \{(\rho, z) \mid \rho \geq 0\}$ with an interval $I = [-m_1, m_1]$ removed from the z-axis. The parameter m_1 is uniquely determined by the initial data.

An Example: Schwarzschild

In Weyl coordinates the Schwarzschild metric is given by

$$U_0 = \frac{1}{2}\log\frac{\sqrt{\rho^2+(z+m)^2}+\sqrt{\rho^2+(z-m)^2}-2m}{\sqrt{\rho^2+(z+m)^2}+\sqrt{\rho^2+(z-m)^2}+2m},$$

$$\alpha_0 = \frac{1}{2}\log\frac{\left(\sqrt{\rho^2+(z+m)^2}+\sqrt{\rho^2+(z-m)^2}\right)^2-4m^2}{4\sqrt{\rho^2+(z+m)^2}\sqrt{\rho^2+(z-m)^2}},$$

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Note that both U_0 and α_0 blow up like $\log\rho$ near the horizon rod, but are smooth and bounded everywhere else. For general initial data the asymptotics will be the same near the horizon, and thus we decompose general data into a Schwarzschid part plus a smooth and bounded part: $U = U_0 + \overline{U}$, $\alpha = \alpha_0 + \overline{\alpha}$.

The structure of the metric in Weyl coordinates implies a nice formula for the scalar curvature

$$2e^{-2U+2\alpha}R = 8\Delta U - 4\Delta_{\rho,z}\alpha - 4|\nabla U|^2 - \rho^2 e^{-2\alpha}\left(\partial_z A_\rho - \partial_\rho A_z\right)^2,$$

where $\Delta_{\rho,z} = \partial_\rho^2 + \partial_z^2$.

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where $\Delta_{\rho,z} = \partial_\rho^2 + \partial_z^2$. Moreover, if $J(\eta) = 0$ and M^3 is simply connected, then a twist potential exists

$$dv = \star\left(k(\eta) \wedge \eta\right),$$

and yields a lower bound for the scalar curvature

$$R \geq |k|^2 \geq 2\frac{e^{6U-2\alpha}}{\rho^4}|\nabla v|^2$$

after using the dominant energy condition.

By integrating the scalar curvature we find that

$$m_{adm} \geq \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(|\nabla\overline{U}|^2+\frac{e^{4U}}{\rho^4}|\nabla v|^2\right)dx$$
$$+\frac{1}{4}\int_{-m_1}^{m_1}(\overline{\alpha}(0,z)-2\overline{U}(0,z))dz+m_1.$$

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The volume integral is the reduced harmonic energy of a singular map $\mathbb{R}^3 \to \mathbb{H}^2$ given by $x \to (u(x), v(x))$ where $u = -\log \rho + U$. As before, we may use convexity of the energy along geodesic deformations in order to minimize this functional and show that its minimum is achieved at the Kerr harmonic map which possesses the same angular momentum and parameter m_1, that is

$$\int_{\mathbb{R}^3} \left(|\nabla \overline{U}|^2 + \frac{e^{4U}}{\rho^4} |\nabla v|^2 \right) dx \geq \int_{\mathbb{R}^3} \left(|\nabla \overline{U}_k|^2 + \frac{e^{4U_k}}{\rho^4} |\nabla v_k|^2 \right) dx.$$

The rod integral may be estimated with maximum principle techniques to yield a similar style lower bound

$$\int_{-m_1}^{m_1}(\overline{\alpha}(0,z)-2\overline{U}(0,z))dz \geq \int_{-m_1}^{m_1}(\overline{\alpha}_k(0,z)-2\overline{U}_k(0,z))dz.$$

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We then have the following result.

Theorem

(MK, Weinstein) Let (M^3,g,k) be a simply connected, axisymmetric, maximal, asymptotically flat initial data set with connected minimal surface boundary, which has nonnegative energy density $\mu \geq 0$ and compatibility condition for the existence of a twist potential $J(\eta)=0$. Then

$$m_{adm} \geq \sqrt{\frac{m_1^2+\sqrt{m_1^4+4\mathcal{J}^2}}{2}},$$

and equality occurs if and only if the data are isometric to the canonical slice of a Kerr spacetime.

The right-hand side of the inequality gives the correct form for the Penrose inequality with angular momentum, but the area is not necessarily that of the initial data

$$m_{adm} \geq \sqrt{\frac{A_k}{16\pi} + \frac{4\pi\mathcal{J}^2}{A_k}}.$$

Here A_k represents the area of the Kerr black hole having angular momentum $\mathcal{J}$ and parameter m_1.

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Question 1: If the boundary is area outermimizing, does this inequality imply the Penrose inequality with angular momentum?

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Question 1: If the boundary is area outermimizing, does this inequality imply the Penrose inequality with angular momentum?

Question 2: How does the parameter m_1 evolve in time? If it is nondecreasing, like the area of the event horizon, then the inequality of the theorem is implied by cosmic censorship.

Mass-Angular Momentum-Charge Inequality

Theorem

(Dain, Chrusciel, Costa, Schoen, Zhou) Let (M^3, g, k, E, B) be a complete, simply connected, axially symmetric, maximal initial data set with one end asymptotically flat and the other either asymptotically flat or asymptotically cylindrical. If $J_{EM}(\eta) = 0$ and the charged dominant energy condition holds then

$$m_{adm}^2 \geq \frac{q^2 + \sqrt{q^4 + 4\mathcal{J}^2}}{2}.$$

Equality holds if and only if the initial data are isometric to the canonical slice of an extreme Kerr-Newman black hole.

Mass-Angular Momentum Inequality

Theorem

(Alaee, MK, Kunduri) Let (M^4, g, k) *be a complete, simply connected with* $H_2(M^4) = 0$*, bi-axially symmetric, maximal initial data set for the 5-dimensional Einstein equations satisfying* $\mu \geq 0$ *and* $J(\eta_i) = 0$*,* $i = 1, 2$ *and with two ends, one designated asymptotically flat and the other either asymptotically flat or asymptotically cylindrical, then*

$$m^3 \geq \frac{27\pi}{32} \left(|\mathcal{J}_1| + |\mathcal{J}_2|\right)^2 .$$

Moreover if $\mathcal{J}_i \neq 0$*,* $i = 1, 2$*, then equality holds if and only if the data set is the canonical slice of an extreme Myers-Perry spacetime.*

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(Alaee, MK, Kunduri) Let (M^4, g, k) be a complete, simply connected with $H_2(M^4) = 0$, bi-axially symmetric, maximal initial data set for the 5-dimensional Einstein equations satisfying $\mu \geq 0$ and $J(\eta_i) = 0$, $i = 1, 2$ and with two ends, one designated asymptotically flat and the other either asymptotically flat or asymptotically cylindrical, then

$$m^3 \geq \frac{27\pi}{32} \left(|\mathcal{J}_1| + |\mathcal{J}_2|\right)^2 .$$

Moreover if $\mathcal{J}_i \neq 0$, $i = 1, 2$, then equality holds if and only if the data set is the canonical slice of an extreme Myers-Perry spacetime.

If one or more angular momenta vanish, then the corresponding extreme Myers-Perry solution either has a naked singularity or is not complete, and thus equality cannot be achieved under the hypotheses of the theorem.