The (in)stability of anti-de Sitter spacetime—resonant approximation

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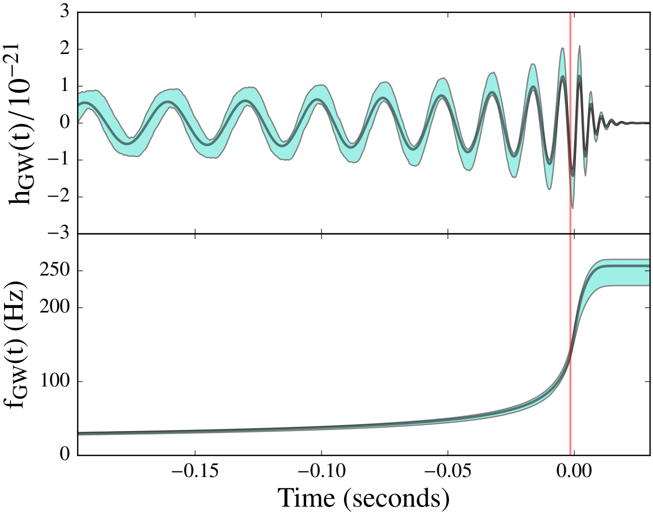
joint work with Piotr Bizoń and Andrzej Rostworowski

Unterstützt von / Supported by

Alexander von Humboldt
Stiftung / Foundation

GR21, New York City, 14 July 2016



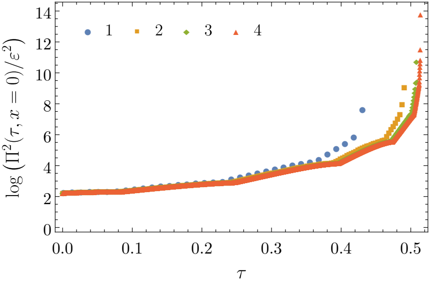
Tests of general relativity with GW150914 [Abbott et al., 2016]

Conjecture (Bizoń-Rostworowski '11)

1. *AdS$_{d+1}$ ($d \geq 3$) is unstable against black hole formation under arbitrarily small perturbations.*
2. *There are perturbations for which turbulent energy transfer is not active (time- and quasi-periodic solutions).*

Einstein-Klein-Gordon system with $\Lambda < 0$ in spherical symmetry

$$\phi(0,x) = \varepsilon\left(\tfrac{1}{4}e_0(x) + \tfrac{1}{6}e_1(x)\right), \quad \dot{\phi}(0,x) = 0, \quad \varepsilon \sim 2^{-p},$$



Extrapolation (here $d = 4$)

$$\tau_H := \lim_{\varepsilon \to 0} \frac{t_H}{\varepsilon^2} \approx 0.514,$$

Turbulent cascade

$$E_j \sim j^{-2}, \quad j \gg 1,$$

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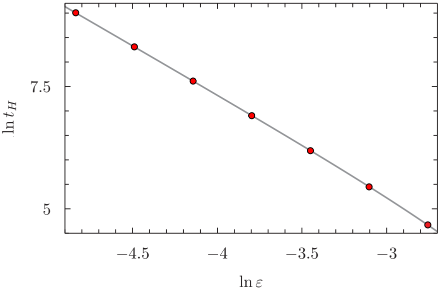

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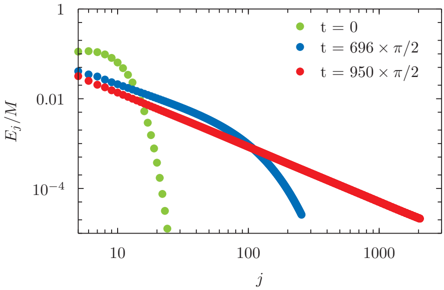

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Linear scalar perturbations of AdS

- Linear equation on an AdS background **[Ishibashi and Wald, 2004]**

$$\ddot{\phi} + \hat{L}\phi = 0\,, \quad \hat{L} = -\frac{1}{\tan^{d-1}x}\,\partial_x\left(\tan^{d-1}x\,\partial_x\right)$$

  This operator is essentially self-adjoint on $\mathcal{H} = L^2\left([0,\pi/2);\tan^{d-1}x\ \mathrm{d}x\right)$.

- Eigenvalues and eigenvectors of essentailly self-adjoint operator $\hat{L}$ on Hilbert space $\mathcal{H}$ are $(j = 0, 1, \ldots)$

$$\omega_j^2 = (d+2j)^2, \quad e_j(x) = \mathcal{N}_j\ \cos^d x\, P_j^{(d/2-1,d/2)}(\cos 2x)\,,$$

- Nondispersive spectrum—nonlinear coupling of linear modes—resonances. Eigenfrequencies are *nondispersive* when $\mathrm{d}\omega_j/\,\mathrm{d}j = \mathrm{const}$, and *dispersive* in the opposite case.
- *Completely* (*fully*) resonant when ω_j rational multiples of one another (*suggesting a large number of secular terms in non-linear perturbation theory*). Resonant spectrum is also defined as: $\exists\ \{k_i|\, k_i \in \mathbb{N}\},\ \sum_i k_i\omega_i = 0$.
- Appearance of resonances should be attributed to the structure of equations not only to the frequency spectrum.

Resonant approximation

- Failure of naïve perturbative approach

$$\phi(t,x) = \varepsilon\,\phi_1(t,x) + \varepsilon^3\phi_3(t,x) + \dots,$$

  Secular terms $\phi(t,x) = \varepsilon\,\phi_1(t,x) + \varepsilon^3\, t(\,\dots) + \dots$
- Resummation (with *slow time* $\tau = \varepsilon^2 t$ dependence)

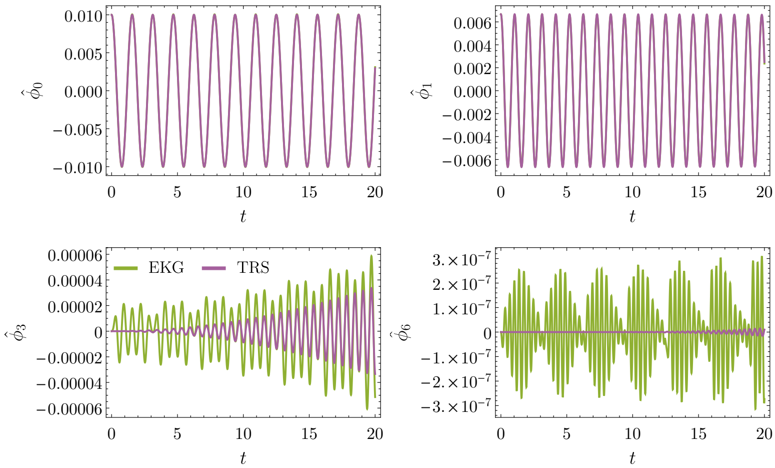
$$\phi_1(t,x) = \sum_{j\geq 0} \left(\alpha_j(\tau)e^{-i\omega_j t} + \bar{\alpha}_j(\tau)e^{i\omega_j t}\right) e_j(x),$$

- Resonant system ($\omega_i \pm \omega_j \pm \omega_k = \pm\omega_m$)

$$\alpha_m' = -\frac{i}{2\omega_m}\left(\sum_{ijk}^{-++} C_{ijkm}^{-++}\alpha_i\alpha_j\bar{\alpha}_k + \sum_{ijk}^{+--} C_{ijkm}^{+--}\bar{\alpha}_i\bar{\alpha}_j\alpha_k + \sum_{ijk}^{+++} C_{ijkm}^{+++}\alpha_i\alpha_j\alpha_k\right),$$

  Invariant under: $\alpha_m(\tau) \to \varepsilon^{-1}\alpha_m(\tau/\varepsilon^2)$.
- Multiscale [Balasubramanian *et al.*, 2014] , renormalization group [Craps *et al.*, 2014] ($C^{+--} = 0$ and $C^{+++} = 0$ for $+--$ and $+++$) and averaging [Craps *et al.*, 2015] approaches. Slow long-time energy flow between the modes.
- We have shown that [Bizoń,M&Rostworowski, 2015]
  - this infinite system has a solution that becomes singular in finite time,
  - singular solution governs generic blowup,

Qualitative comparison



$$\phi(t,x) = \sum_{j\geq 0} \hat{\phi}_j(t) e_j(x), \quad \hat{\phi}_j(t) \approx \varepsilon\, 2\, \Re\left(\alpha_j\left(\varepsilon^2 t\right) e^{-i\omega_j t}\right),$$

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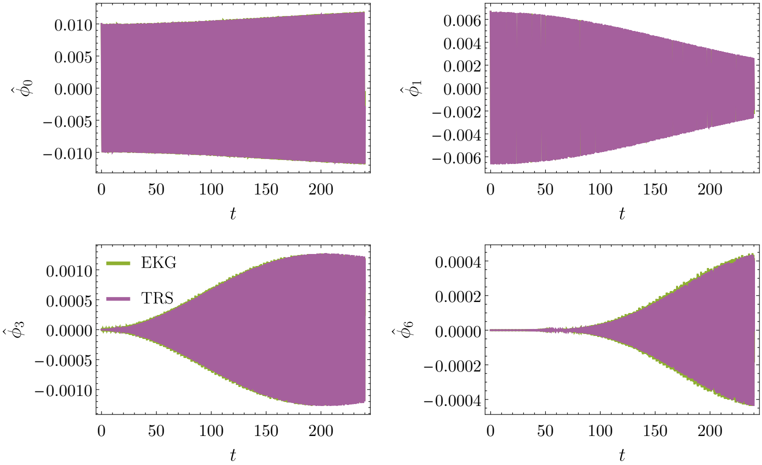

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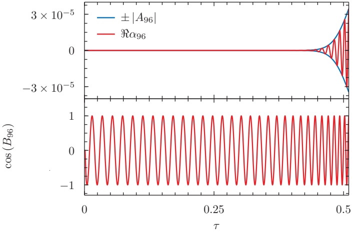
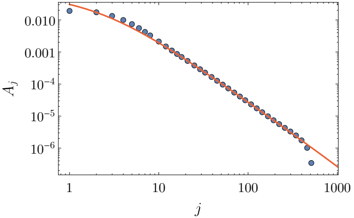
Resonant approximation–blowup





- Universal behavior $\alpha_m(\tau) = A_m(\tau)e^{iB_m(\tau)}$

$$A_m(\tau) \sim \tau^{m-1}, \quad B_m(\tau) \nearrow,$$

- Asymptotic ansatz ($m \gg 1$)

$$A_m(\tau) \sim m^{-\gamma(\tau)} e^{-\rho(\tau)m},$$

Analyticity strip method [Sulem *et al.*, 1983] , [Bizoń and Jałmużna, 2013] (AdS_3).

- Data fits

$$\gamma(\tau) \approx 2, \quad \rho(\tau) \approx \rho_0(\tau_* - \tau),$$

as $\tau \to \tau_*$ ($\approx 0.513 \approx \tau_H$)

- Solution for phases $B_m(\tau)$?

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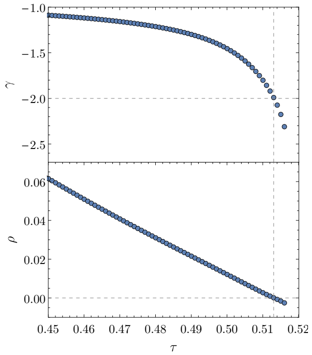

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Resonant approximation–blowup

$$2\omega_m A_m' = \sum_{ijk}{}' S_{ijkm} A_i A_j A_k \Im\left(e^{i\left(B_i+B_j-B_k-B_m\right)}\right),$$

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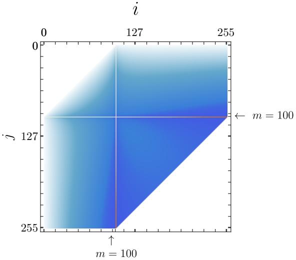

- Synchronized phases **[Freivogel and Yang, 2015]**

$$B_j \sim j,$$

- Dominant contribution

$$R_{jm} \sim j^3 m^2,$$

Then for $A_j \sim j^{-2} e^{-\rho_0(\tau_* - \tau)j}$:

$$\sum_{j\neq m} R_{jm} A_j^2 \sim m^2 \sum_{j\neq m} \frac{1}{j} e^{-2\rho_0(\tau_*-\tau)j} \sim m^2 \log(\tau_* - \tau),$$

yields logarithmic blowup of phases $\tau \to \tau_*$

$$B_m'(\tau) = a_m \log\left(\tau_* - \tau\right) + b_m.$$

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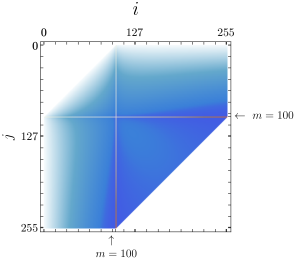

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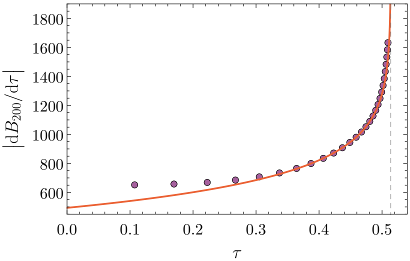

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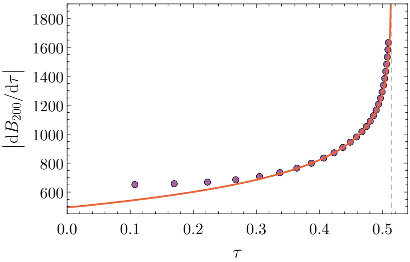

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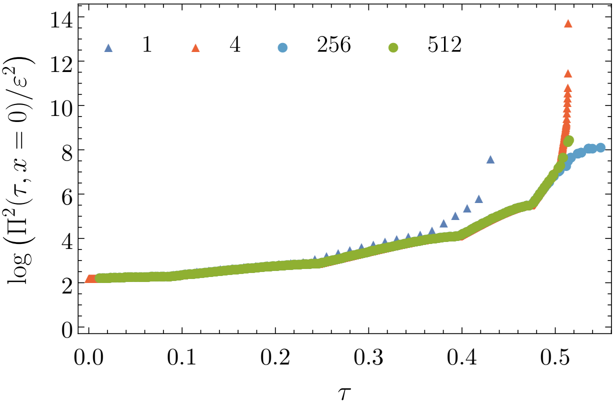
Resonant approximation–blowup and collapse



Convergence with $\varepsilon \to 0$ (Einstein's equation), and for $N \to \infty$ (truncated resonant system).

Resonant approximation

- Method intended to provide uniformly bounded solution gives hints for instability. Note $|\alpha_j| < \infty \not\Rightarrow |\alpha_j'| < \infty$, here $(d = 4)$

$$|\alpha_j| < \infty, \quad \text{but} \quad |\alpha_j^{(k)}| \sim \left(\frac{1}{\tau_* - \tau}\right)^{k-1}, \ \ k \geq 1,$$

- Generalization of asymptotic $\tau \to \tau_*$ solution to $d \geq 4$

$$A_j \sim j^{\gamma} e^{-\rho j}, \quad \gamma \to -d/2, \quad \rho \to 0,$$

which blows up in finite time τ_*. The character of blowup is oscillatory, i.e. phases behave as (in the interior gauge)

$$B_j'(\tau) \sim \log(\tau_* - \tau), \quad \text{for} \quad d \geq 4.$$

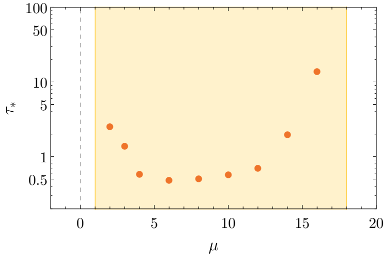
In boundary gauge blowup of higher derivatives. The $d = 3$ case?

- Energy spectra $|\alpha_j| \sim j^{-d/2} \ \Rightarrow \ E_j \sim j^{2-d}$. Dimensional argument [Bizoń and Rostworowski, 2012?] , also [Freivogel and Yang, 2015] .
- But not all data leads to unbounded growth of higher Sobolev type norms.

Resonant approximation–two-mode initial data

$$\phi(0,x) = \varepsilon(\mu)\left(\frac{1}{\omega_0}e_0(x) + \frac{1}{\mu}e_1(x)\right), \quad \dot{\phi}(0,x) = 0,$$

Stability islands (of time-periodic solutions [M&Rostworowski, 2013] , [Kim, 2015] , [Fodor *et al.*, 2015] ?)



- $\mu \to \infty$, $\alpha_0(\tau) = \varepsilon e^{-i\frac{T_0}{2\omega_1}\tau}$, $\alpha_{j\neq 0} = 0$
- $\mu \to 0$, $\alpha_1(\tau) = \varepsilon e^{-i\frac{T_1}{2\omega_1}\tau}$, $\alpha_{j\neq 1} = 0$
- Stationary solutions $|\alpha_j| = \mathrm{const}$

$$\alpha_j(\tau) = A_j e^{iB_j\tau},\ B_j = c_1 j + c_2,$$

Construction [Balasubramanian *et al.*, 2014] , stability [Green *et al.*, 2015] and asymptotics [Craps *et al.*, 2015] . What is their role in dynamics of generic initial conditions?

The same story for *wide gaussians* [Buchel *et al.*, 2013] , [M&Rostworowski, 2013] .

Conclusions and questions

- Extend study of the resonant system. How to transfer blowup to the full system? How to interpret oscillatory singularity? Choptuik's critical solution [Choptuik, 1993] ?
- Nontrivial (*complicated*) phase-space of solutions to the Einsteins equation with negative cosmological constant. How large the islands of stability are? Understand the role of stationary solutions in the dynamics [Green *et al.*, 2015] .
- Do we understand borderline between collapse and quasiperiodic motion?
- Structure of resonances [Craps et al., 2014, 2015] and their impact on nonlinear evolution; relaxed symmetry assumption–work in progress.
- Clash between different numerical approaches ([Balasubramanian et al., 2014] and [Bizoń and Rostworowski, 2014]) shows that long-time evolution of asymptotically AdS solutions is particularly demanding.
- Weak turbulence–common for (non-integrable) NWE on bounded domains (NLS on torus [Colliander *et al.*, 2010] , [Carles and Faou, 2012]).
- Challenging mathematical problems, both for any attempts to rigorous proofs and numerical analysis. Meeting point of GR, theory of PDEs, turbulence, and HEP which makes it an exciting field of research.