The Pennsylvania State University
Institute for Gravitation & the Cosmos

Entanglement production and Lyapunov exponents

GR21, New York City, 2016

Session D4: Quantum fields in curved space-time, etc.

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work with Eugenio Bianchi and Nelson Yokomizo

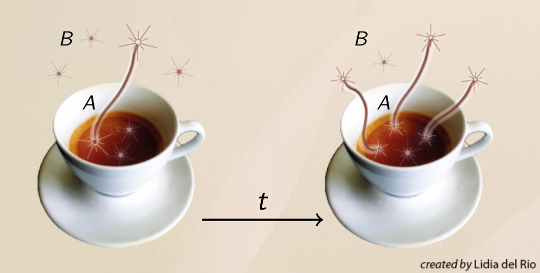
[arXiv:1507.01567; arXiv:1512.08959; arXiv:1607.xxxxx]

Subject: Time evolution of entanglement entropy

Choice of State:
$|\psi(t)\rangle \in \mathcal{H}$

Choice of Subsystem:
A and B

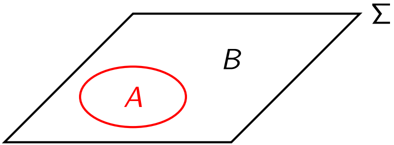
Entanglement entropy:
$S_A(t) := S_A(|\psi(t)\rangle)$



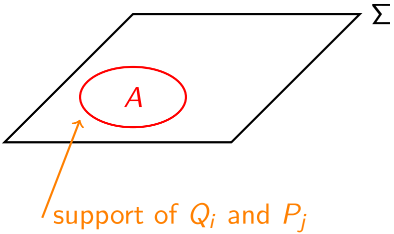
Subsystems in QFTs

Choices of a subsystem for a QFT with $(\hat{\varphi}(x), \hat{\pi}(x))$:

- **Spatial region $A \subset \Sigma$?**
  We can't split Hilbert space as
  $$\mathcal{H}_{\mathrm{QFT}} = \mathcal{H}_A \otimes \mathcal{H}_B \,,$$
  because Fock space is *non-local*.
  [Reeh, Schlieder]



- **"Detector"**
  Subsystem with finite N_A DOFs
  $$\hat{q}_i = \int Q_i(x)\,\hat{\varphi}\,d^3x$$
  $$\hat{p}_j = \int P_j(x)\,\hat{\pi}\,d^3x$$
  with weight functions Q_i and P_i,
  such that $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$.
  [Balachandran et al.]



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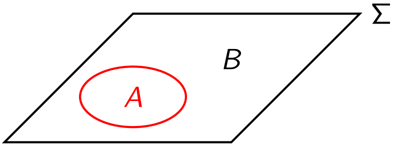

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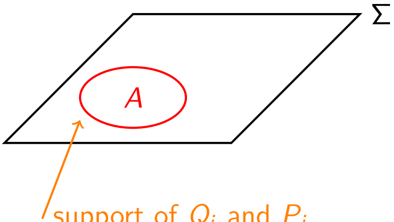

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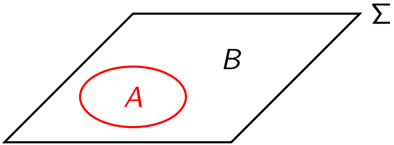

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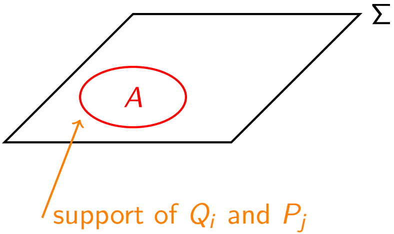
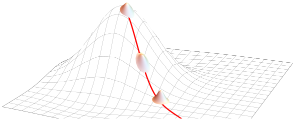
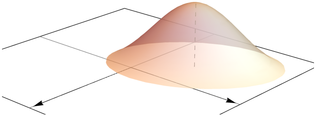


Instabilities lead to linear entanglement production:



(1) Systems with Instabilies

$\Rightarrow$ Lyapunov exponents λ_i

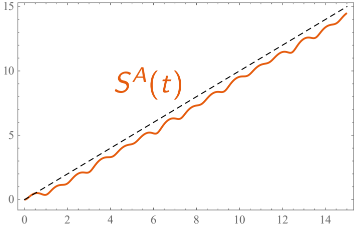


(2) Gaussian initial state

$\Rightarrow$ squeezed coherent $|\zeta, g\rangle$

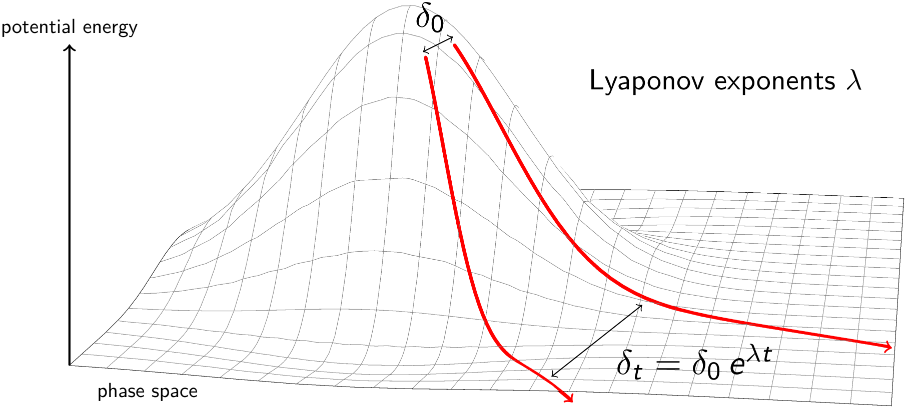
Linear entanglement production

$$S^A(t) \sim \Lambda_A\, t$$



Instability = Sensitivity to initial conditions:

Initial deviation δ_0 grows exponentially to $\delta_t = \delta_0\, e^{\lambda t}$



(1) Classical instabilities

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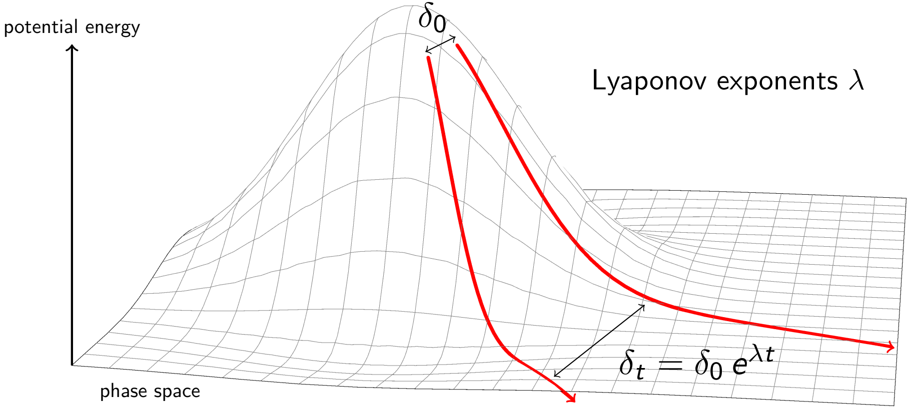


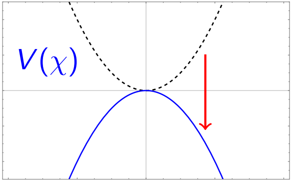
Uncertainty grows with Kolmogorov-Sinai entropy rate: $S_{KS} = \sum_{\lambda_i > 0} \lambda_i$

(1) Instabilities in QFTs

Let us consider a scalar quantum field on homogeneous slice Σ.
Two examples of instabilities are:

- **Upside-down potential (Inflation)**
  Scalar field coupled to inflating FLRW

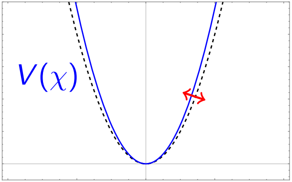
$$\ddot{\chi}_k + \underbrace{\left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right)}_{\text{negative for some modes}} \chi_k = 0$$



  $\Rightarrow$ Mode asymptotics: $\chi_k(\eta) \sim e^{\pm\lambda\eta}$.

- **Parametric resonance (Preheating)**
  Scalar field coupled to inflaton field Φ

$$\ddot{\varphi}_k + \underbrace{\left(k^2 + m_\chi^2 + g^2\Phi^2 \sin^2(mt)\right)}_{\text{resonant for some modes}} \varphi_k = 0$$



  $\Rightarrow$ Mode asymptotics: $\varphi_k(\eta) \sim e^{\pm\mu t}$

[Allahverdi, Brandenberger, Cyr-Racine, Mazumdar]

(1) Our model systems

System (N DOFs):
Quantum system on linear phase space

$$\xi^a = (\varphi_i, \pi_j) \in V = \mathbb{R}^{2N}$$

with symplectic form Ω.

We recover the full QFT in the limit $N \to \infty$.

Subsystems (N_A and N_B DOFs):
We consider Hilbert space decompositions

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

induced by phasespace splitting $V = A \oplus B$ into symplectic subspaces A and B.

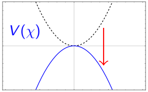
N_A stays fixed, while $N_B \to \infty$, when taking the limit $N \to \infty$. Subsystem is well defined, even if there is no tensor product structure.

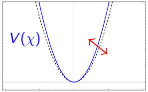
Time evolution:
Quadratic Hamiltonian (can depend on t)

$$H(t) = \frac{1}{2} h(t)_{ab}\, \xi^a \xi^b$$

induces Hamiltonian flow $M(t) : V \to V$.

Example:
Upside-down & Resonance

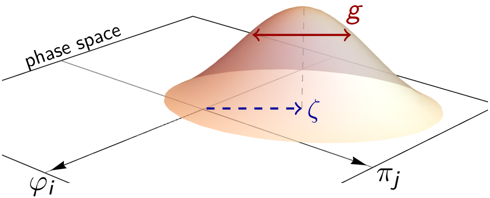




(2) Gaussian initial states

Gaussian (= squeezed coherent) states $|\zeta, g\rangle$ can be characterized by

- Expectation value (peak)
  $\zeta^a = \langle \hat{\xi}^a \rangle$
- Correlation metric (spread)
  $g^{ab} = \frac{1}{2}\langle \{ \hat{\xi}^a, \hat{\xi}^b \} \rangle - \zeta^a \zeta^b$
  Complex structure: $J = \Omega g$



Entanglement entropy for symplectic subsystem $A \subset V$

Restricting complex structure $[J]_A$ to subsystem A gives:

$$S^A(|\zeta, g\rangle) = \mathrm{Tr}\left(\frac{\mathbb{1} - i[J]_A}{2}\right) \log \left| \frac{\mathbb{1} - i[J]_A}{2} \right|$$

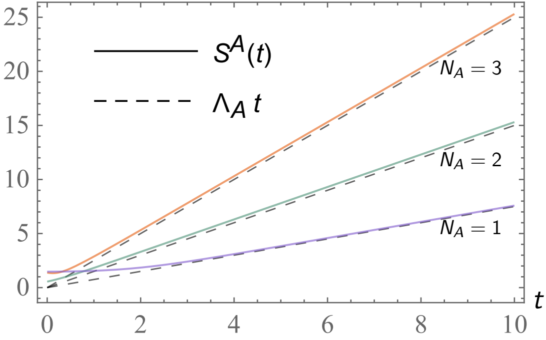
Note: This is the same complex structure J that one uses in QFT in curved spacetime to choose a Fock space representation by decomposing fields into positive and negative frequency modes.

Linear entanglement production theorem

Unstable quadratic Hamiltonians + Gaussian initial state imply:

$$S^A(t) \sim \Lambda_A\, t \quad \text{with} \quad \Lambda_A = \sum_{i=1}^{2N_A} \lambda_i \quad (\text{largest } 2N_A \text{ Lyaponov exponents } \lambda_i)$$

Proof in [Bianchi, LH, Yokomizo], related work by [Berenstein, Asplund]

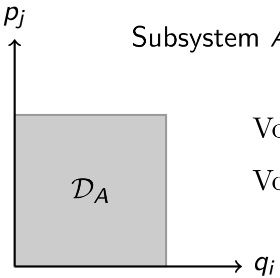


Step 1: Geometric entanglement entropy

For highly entangled systems, we find the asymptotics

$$S^A(|\zeta, g\rangle) \sim \log \mathrm{Vol}_g(\mathcal{D}_A)\,,$$

where we take a Darboux region $\mathcal{D}_A \subset A$ with *symplectic volume* 1 and compute its *metric volume* with respect to g (restricted to A).

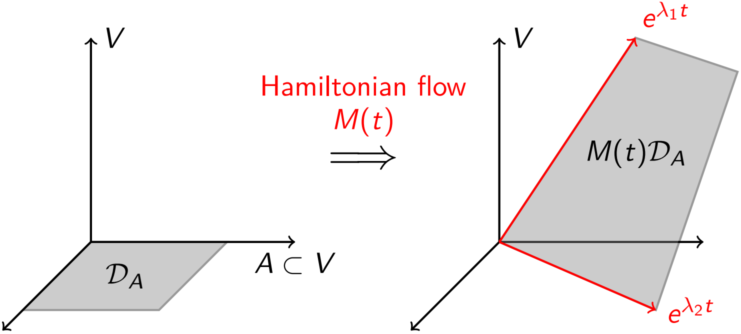


$\mathrm{Vol}_\Omega(\mathcal{D}_A) = 1$ *(symplectic volume)*

$\mathrm{Vol}_g(\mathcal{D}_A) \geq 1$ *(metric volume)*

Step 2: Exponential stretching at Instabilities
We can compute the classical solution as symplectic flow $M(t): V \to V$.
The entanglement entropy of the time evolved Gaussian states $|\zeta_t, g_t\rangle$ is

$$\boxed{S^A(t) \sim \log \mathrm{Vol}_{g_t}(\mathcal{D}_A) = \log \mathrm{Vol}_{g_0}(M(t)\mathcal{D}_A) \sim \sum_{i=1}^{2N_A} \lambda_i t\,.}$$



Summary

Entropy of a "Detector" probing N_A degrees of freedom

Unstable quadratic Hamiltonians + squeezed coherent initial state imply:

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Important properties:

- Rate is independent of initial state $|\zeta_0, g_0\rangle$
- Rate only depends on the dimension $\dim A = 2N_A$ of subsystem
- Lyapunov exponents come in opposite pairs ($\lambda_i = -\lambda_{2N-i+1}$):
  $\lambda_1 \geq \cdots \geq \lambda_N \geq 0 \geq \lambda_{N+1} \geq \cdots \geq \lambda_{2N}$
- Upper bound for *all* initial states

Connection to Kolmogorov-Sinai:

This is equal to the Kolmogorov-Sinai entropy $S_{KS} = \sum_{\lambda_i>0} \lambda_i$ for sufficiently large N_A, where N_A is the number of DOFs of the detector.