The threefold classification of spacetime tensors

Lode Wylleman

Department of Mathematical Analysis, Ghent University, Belgium

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Very general result

for tensors over spacetime (M, g)

- arbitrary dimension d
- arbitrary Lorentz metric g
- pointwise $(V = T_pM)$
- arbitrary tensors $T \neq 0$ (covariant, valence m)

Main new result

to a *generic* tensor a unique timelike direction is associated, endowing T_pM with a $1 + (d - 1)$ structure

Tensor classification: a non-exhaustive history

- Petrov 1954: Ricci (general d + signature!) + Weyl ($d = 4$)
- Synge 1956: Maxwell
- null alignment classification
  - Weyl, $d = 4$: Debever 1959, Bel 1959, Penrose 1960
  - Weyl, arbitrary d: Coley, Milson, Pravda, Pravdova 2004
  - arbitrary T and d: Milson, Coley, Pravda, Pravdova 2005

Null alignment = lowest boost order classification

- consider fixed T and arbitrary null direction $[k]$
- take any ℓ along $[k]$ and any null frame of the form

$$(m_\alpha)_{\alpha=0}^{n-1} = (m_0 = \ell, m_1 = n, m_i), \quad \ell \cdot n = m_i \cdot m_i = 1$$

- boost weight under boost $\ell \mapsto e^{\lambda}\ell,\ n \mapsto e^{-\lambda}n,\ m_i \mapsto m_i$:

$$T_{\alpha_1\cdots\alpha_m} \mapsto e^{\lambda b_{\alpha_1\cdots\alpha_m}} T_{\alpha_1\ldots\alpha_m}, \quad b_{\alpha_1\cdots\alpha_m} = \sum_{i=1}^{m} (\delta_{\alpha_i 0} - \delta_{\alpha_i 1})$$

- boost order $b(k)$ = maximum $b_{\alpha_1\cdots\alpha_m}$ for $T_{\alpha_1\cdots\alpha_m} \neq 0$
- (primary) alignment type of T = minimum $b(k)$

BOZANDs and 3-fold classification

$$b_{\max} := \max_{\{k\}} b(k)$$

$[k]$ is

- AND (aligned null direction) of T if $b(k) < b_{\max}$
- BOZAND (AND of boost order zero or less) if $b(k) \leq 0$

$d = 4$ Weyl: AND = PND (principal null direction)
BOZAND = multiple PND

3-fold classification

$\#$ BOZANDs	general type	Weyl: alignment type
0	A	G or I
1	B	II or III or N
≥ 2 (null dirs of $\mathcal{L}_n$)	D	D

Extended alignment theorem

Alignment theorem (Hervik 2011)

T characterized by polynomial invariants $\Leftrightarrow$ type A or D

future-pointing unit timelike vector u (observer at p)

- reflection about $u^\perp$: θ_u
- Euclidean product: $\langle S, T \rangle_u := S \cdot \theta_u(T)$
- super-energy density (Senovilla 2000): $W(u) = \langle T, T \rangle_u$

Extension (Hervik-Ortaggio-Wylleman 2013)

type A or D $\Leftrightarrow$ the map $u \mapsto W(u)$ reaches a minimum

Type A/D: how many 'minimal' observers?

= observers u minimizing $W(u)$

inspiration from $d = 4$ Weyl

- Petrov type D: all observers lying in $\mathcal{L}_0$
- Petrov type I: unique observer (cf. 'transversal frame')

Main theorem

If T is of type
(i) A then there is a unique minimal observer,
(ii) B then there is no minimal observer,
(iii) D then the minimal observers are those lying in $\mathcal{L}_n$.

general type	# BOZANDs	# min. obs.
A	0	1
B	1	0
D	≥ 2 (null dirs. of $\mathcal{L}_n$)	∞ (obs. of $\mathcal{L}_n$)

Minimality: covariant criterion

$$\theta_u \equiv \theta^a_b = \delta^a_b + 2u^a u_b, \quad h^a_b := \delta^a_b + u^a u_b$$

u is minimal iff

$$P^a := T_{b_1 \cdots b_m} T^{c_1 \cdots c_m} \sum_{i=1}^{m} \theta^{b_1}_{c_1} \cdots u^{b_i} h^a_{c_i} \cdots \theta^{b_m}_{c_m} = 0$$

Check with Ferrando-Saez 2013

- Maxwell tensor: $P^a \propto$ Poynting vector
- Weyl tensor: $P^a \propto$ super-Poynting vector

BOZAND: covariant criterion

- more general: $b(k) = b_0$ can be expressed by set of covariant equations, linear in T and homogeneous in k
- generalizes Bel-Debever criteria for Weyl (Ortaggio 2009)
- principle:

$$
\begin{aligned}
k^a T_{a\cdots} = 0 \quad &\Leftrightarrow \quad T_{0\cdots} = 0 \\
k_{[a} T_{b]\cdots} = 0 \quad &\Leftrightarrow \quad T_{0\cdots} = T_{i\cdots} = 0 \\
T_{a\cdots} = 0 \quad &\Leftrightarrow \quad T_{0\cdots} = T_{i\cdots} = T_{1\cdots} = 0
\end{aligned}
$$

Consequence for orthogonal isotropy group of tensor

Isotropy group of T within G (generalized orthogonal group):

- type A: subgroup of $\mathrm{Fix}(u)$
- type B: subgroup of $\mathrm{Fix}(k)$
- type D: subgroup of $\mathrm{Fix}(\mathcal{L}_n)$

Some applications

- Equivalence problem / Cartan-Karlhede spacetime classification in arbitrary dimensions (result applicable to $\nabla^i R$; difficulties in practice!)
- Find spacetimes of type A with e.g. given properties of ∇u
- type A subtypes based on u-weight
- connection with physical theories described by a tensor

Conclusions

type	Weyl	# min. obs.	# BOZANDs
A	G/I	1	0
B	II/III/N	0	1
D	D	∞ (obs. of $\mathcal{L}_n$)	≥ 2 (null dirs. of $\mathcal{L}_n$)

- A tensor T at a spacetime point p may admit 'minimal' observers, i.e., observers for which the super-energy density of T relative to u attains a minimum. Generically, there is a unique minimal observer (type A). If such an observer does not exist (type B) then there is a unique BOZAND.
- If there is no BOZAND (type A; Weyl: G/I) then there is a unique minimal observer.
- For type D: the BOZANDs and minimal observers are precisely the null directions, respectively, observers lying within a Lorentzian subspace $\mathcal{L}_n$ of T_pM.