Spectral dimension in non–commutative geometry

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Why am I doing this?

MC simulations can measure $\langle f(g) \rangle$, but what are good $f(g)$?

Should be

- completly covariant
- space independent
- efficient to measure
- connect to physics?

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A few examples

- Phase transitions, Critical exponents (Thermodynamics)
- Transition amplitudes between boundary states
- Spectral properties

Non commutative geometry

$$(s, \mathcal{H}, \mathcal{A}, \Gamma, J, \mathcal{D})$$

- Hilbert space
- Algebra
- Dirac operator
- signature
- Chirality
- Real structure

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Classical $(1,3)$d geometry

- ▶ $L^2(\mathcal{M}, S)$ the L^2 spinors
- ▶ Functions $C^\infty(\mathcal{M}) : f_1(x)$
- ▶ $\mathcal{D} = \displaystyle{\not{\partial}}$
- ▶ $s = (q - p) \bmod 8 = 2$
- ▶ "γ^5"
- ▶ charge conjugation

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Conditions on $\mathcal{D}$

- ▶ $\mathcal{D} = \mathcal{D}^{\dagger}$
- ▶ $\mathcal{D}\Gamma = \pm\Gamma\mathcal{D}$
- ▶ $\mathcal{D}J = \pm J\mathcal{D}$ signs depend on s
- ▶ $[[\mathcal{D}, \rho(a)\triangleright], \triangleleft\rho(b)] = 0$: first order condition

Dirac operator : Form

In general

$$\mathcal{D}(v \otimes m) = \sum_i \omega^i v \otimes \big(\overbrace{K_i m}^{\text{left action}} + \epsilon' \overbrace{m K_i^*}^{\text{right action}} \big)$$

For the example of the $(3,1)$ geometry

$$\mathcal{D} = \sum_{j<k=1}^3 \gamma^0 \gamma^j \gamma^k \otimes [L_{jk}, \cdot] + \gamma^1 \gamma^2 \gamma^3 \otimes \{H_{123}, \cdot\}$$

$$+ \gamma^0 \otimes \{H_0, \cdot\} + \sum_{i=1}^3 \gamma^i \otimes [L_i, \cdot]$$

With H hermitian and L anti-hermitian and traceless

The action

$\mathcal{S} = g_2 \text{Tr} \left(\mathcal{D}^2 \right) + \text{Tr} \left(\mathcal{D}^4 \right)$

What do we want from an action?

- physical motivation $\Rightarrow$ lowest order when expanding a heat kernel
- bounded from below $\Rightarrow$ for some g_2, g_4

The action

$$\mathcal{S} = g_2 \mathsf{Tr}\left(\mathcal{D}^2\right) + \mathsf{Tr}\left(\mathcal{D}^4\right)$$

How does this look for a given geometry?

$(2,0)$ geometry

$$\mathcal{D} = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$$

$$\mathrm{Tr}\,\mathcal{D}^2 = 4n(\mathrm{Tr}\,H_1^2 + \mathrm{Tr}\,H_2^2) + 4\left((\mathrm{Tr}\,H_1)^2 + (\mathrm{Tr}\,H_2)^2\right)$$

$$\mathrm{Tr}\,\mathcal{D}^4 = 4n\bigg(\mathrm{Tr}\,H_1^4 + \mathrm{Tr}\,H_2^4 + 4\,\mathrm{Tr}\,H_1^2 H_2^2 - 2\,\mathrm{Tr}\,H_1 H_2 H_1 H_2\bigg)$$
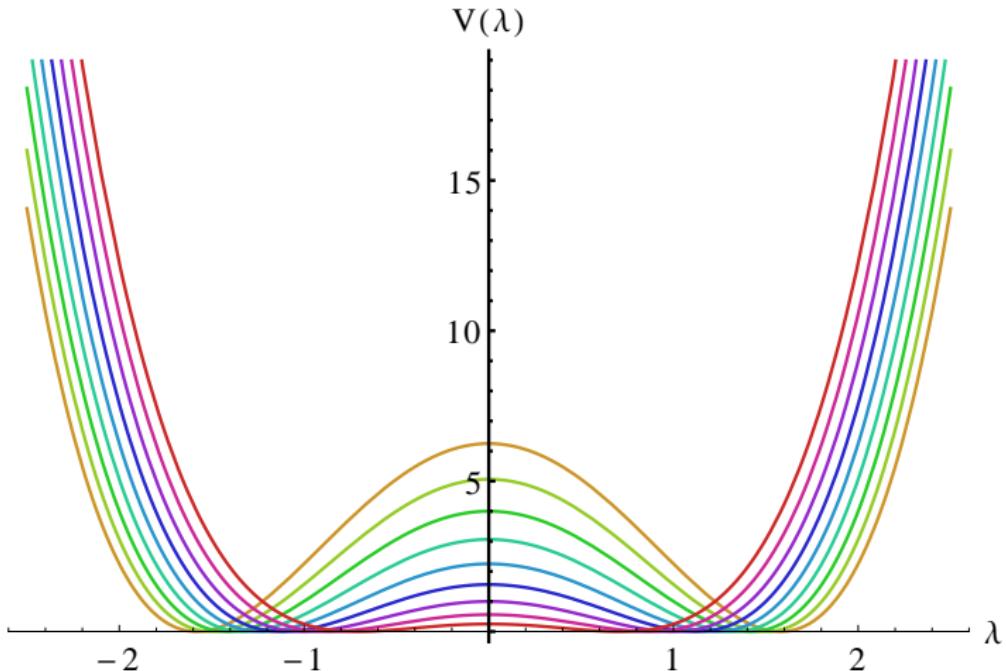
$$+ 16\bigg(\mathrm{Tr}\,H_1\left(\mathrm{Tr}\,H_1^3 + \mathrm{Tr}\,H_2^2 H_1\right)$$

$$+ \mathrm{Tr}\,H_2\left(\mathrm{Tr}\,H_1^2 H_2 + \mathrm{Tr}\,H_2^3\right) + (\mathrm{Tr}\,H_1 H_2)^2\bigg)$$

$$+ 12\bigg((\mathrm{Tr}\,H_1^2)^2 + (\mathrm{Tr}\,H_2^2)^2\bigg) + 8\,\mathrm{Tr}\,H_1^2\,\mathrm{Tr}\,H_2^2$$

The action

$$\mathcal{S} = g_2 \mathsf{Tr}\left(\mathcal{D}^2\right) + \mathsf{Tr}\left(\mathcal{D}^4\right)$$
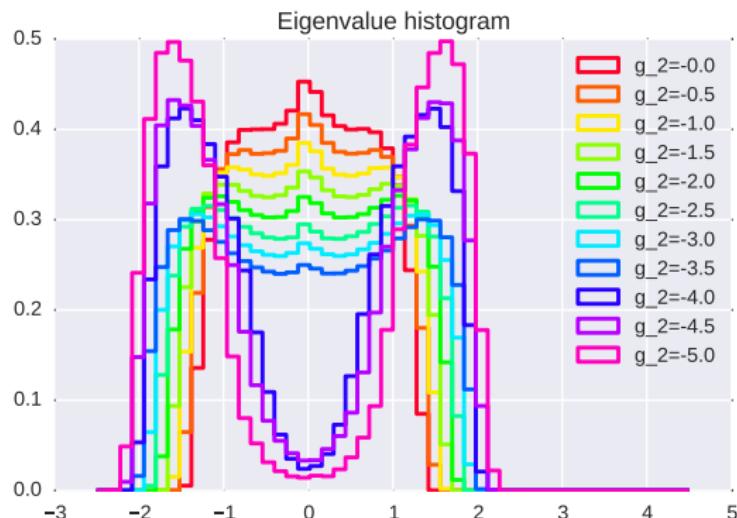
$$g_2 = \{-1, -1.5, \cdots, -4.5, -5\}$$

The action

$$\mathcal{S} = g_2 \mathsf{Tr}\left(\mathcal{D}^2\right) + \mathsf{Tr}\left(\mathcal{D}^4\right)$$

Type (1,3)

Eigenvalue histogram

Legend:
- g_2=-0.0
- g_2=-0.5
- g_2=-1.0
- g_2=-1.5
- g_2=-2.0
- g_2=-2.5
- g_2=-3.0
- g_2=-3.5
- g_2=-4.0
- g_2=-4.5
- g_2=-5.0
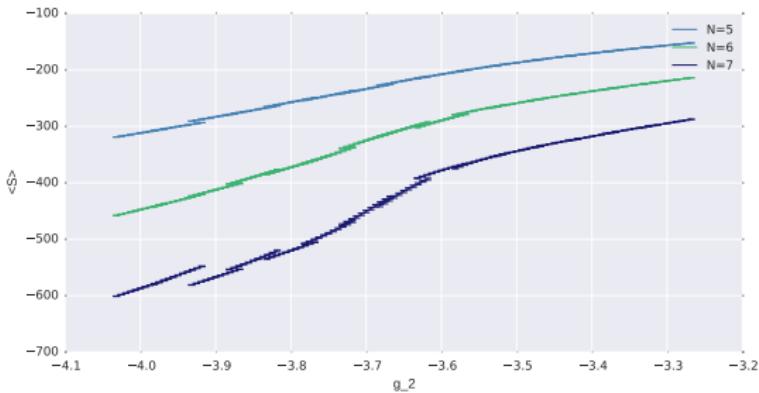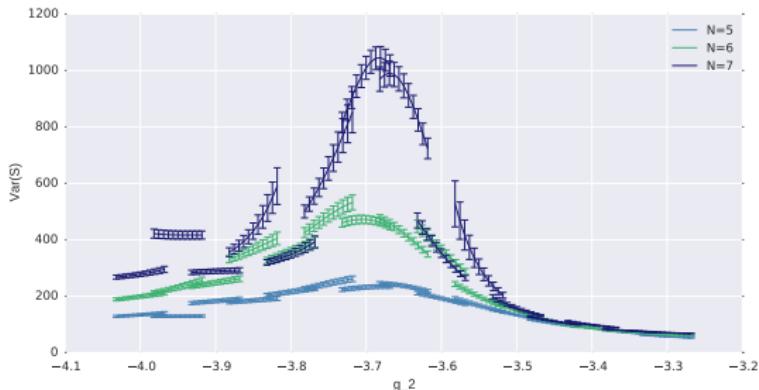
Scaling with size
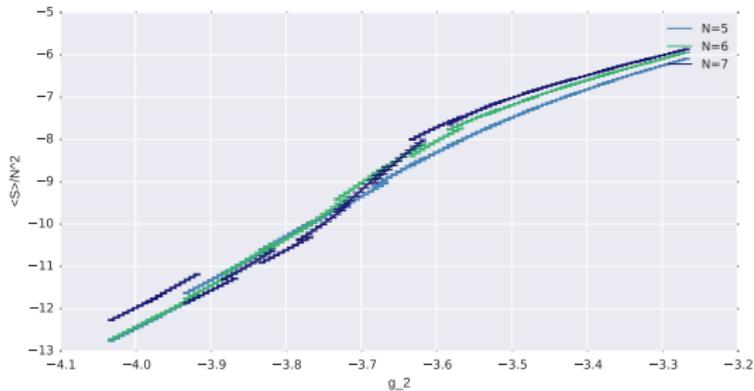
average Action



Variance of the
Action

Scaling with size

average Action
rescaled with N^{-2}

Variance of the
Action
rescaled with N^{-4}
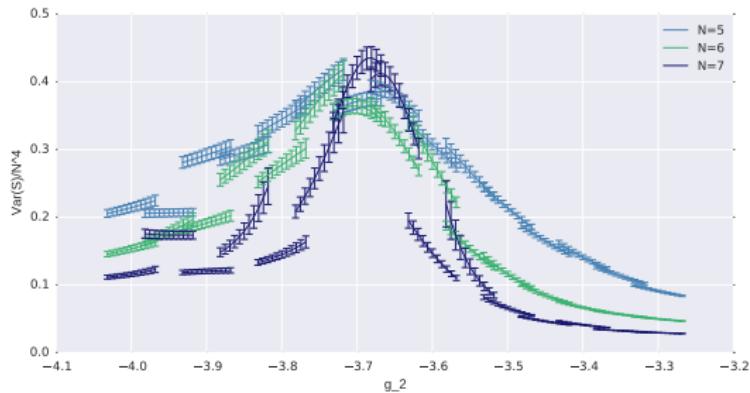
Spectral dimension/ spectral variance

$$P_g(t) = \frac{1}{V} \sum_\lambda e^{-t|\lambda|}$$

Return probability?

Only for Δ, for $\mathcal{D}$ Partition function for ensemble w. energies λ

Spectral dimension/ spectral variance

$$P_g(t) = \frac{1}{V} \sum_\lambda e^{-t|\lambda|}$$

$$D_s(t) = -t \frac{\partial \log[P_g(t)]}{\partial t}$$

Spectral dimension

$\langle t|\lambda| \rangle$ in ensemble of possible e.v. on geometry

Spectral dimension/ spectral variance

$$P_g(t) = \frac{1}{V} \sum_\lambda e^{-t|\lambda|}$$
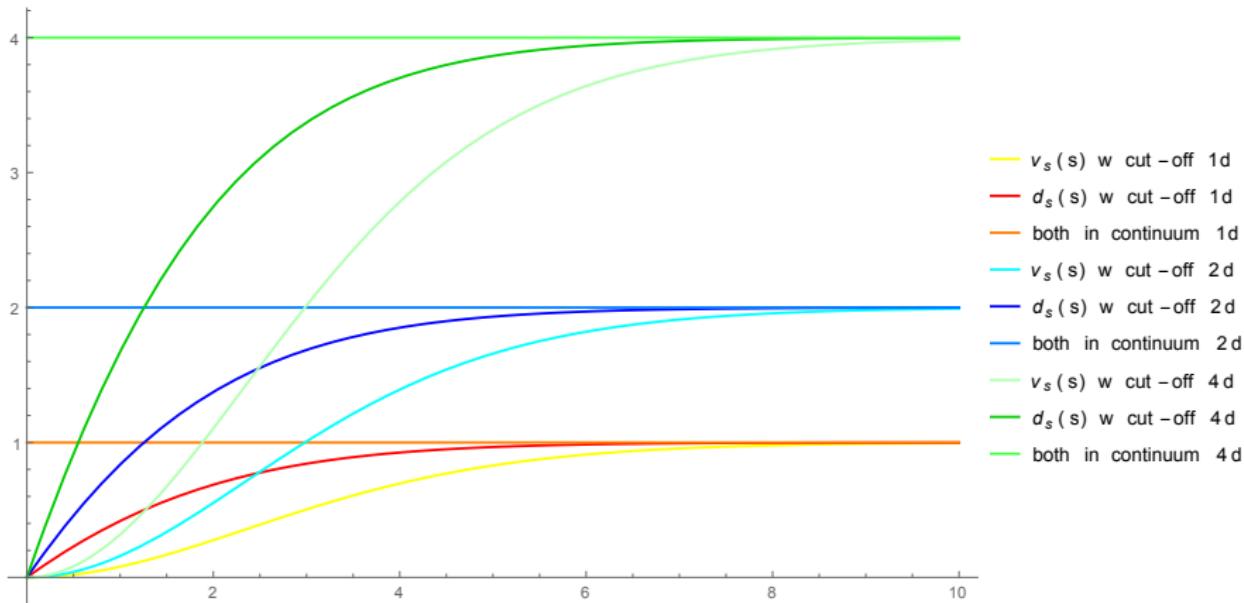
$$D_s(t) = -t \frac{\partial \log[P_g(t)]}{\partial t}$$

$$V_s(t) = D_s(t) - t \frac{\partial D_s(t)}{\partial t}$$

Spectral variance
Variance $\mathrm{Var}(t|\lambda|)$

Flat space

D_s and V_s agree with the ontinuum dimension, and go to 0 when a short distance cut off is introduced.

The sphere: S^2

Legend:
- $d_s(s,5)$
- $v_s(s,5)$
- $d_s(s,10)$
- $v_s(s,10)$
- $d_s(s,100)$
- $v_s(s,100)$
- $d_s(s,\text{continuum})$
- $v_s(s,\text{continuum})$
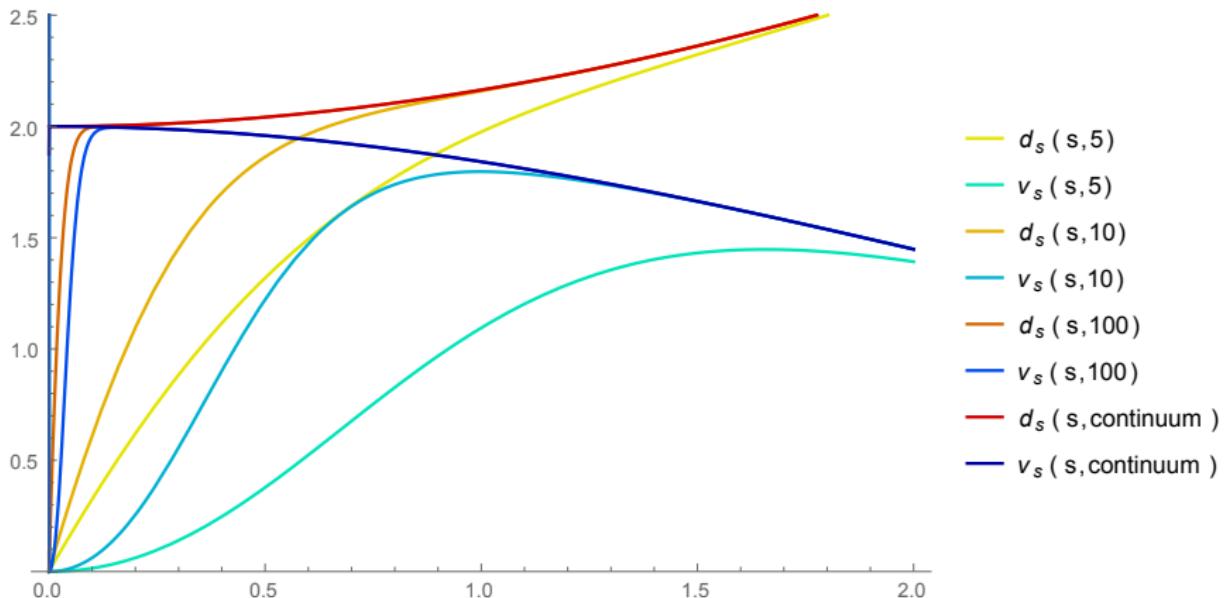
Continuum sphere

D_s goes to infinity when calculated for $\mathcal{D}$, because $\lambda_{min} \neq 0$
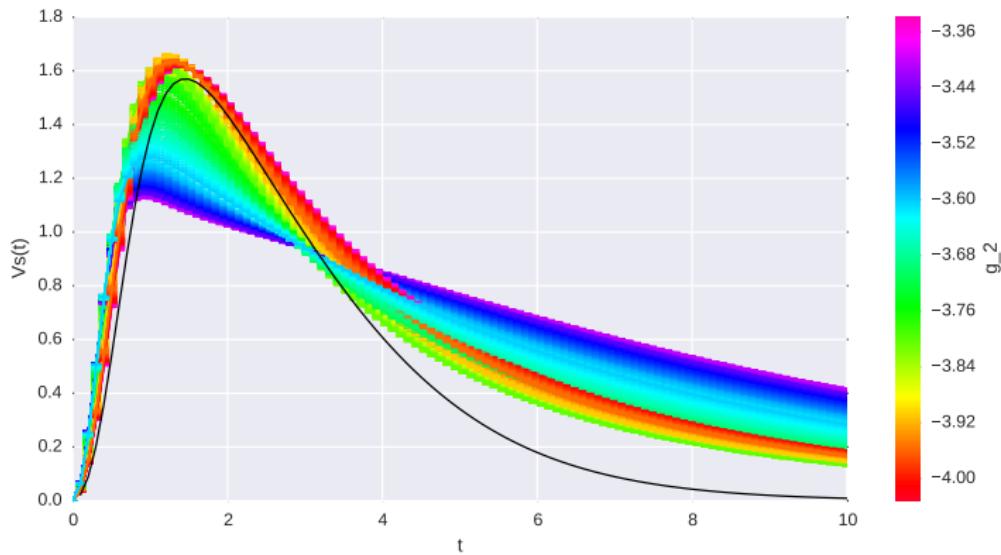
Random Geometry

$$N = 5$$
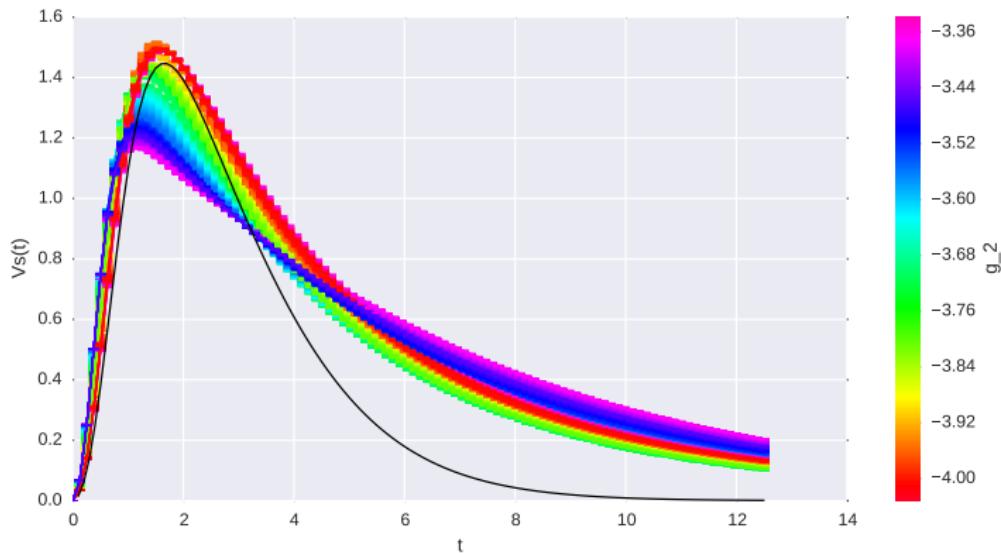
Random Geometry

$$N = 6$$

Random Geometry

$$N = 7$$
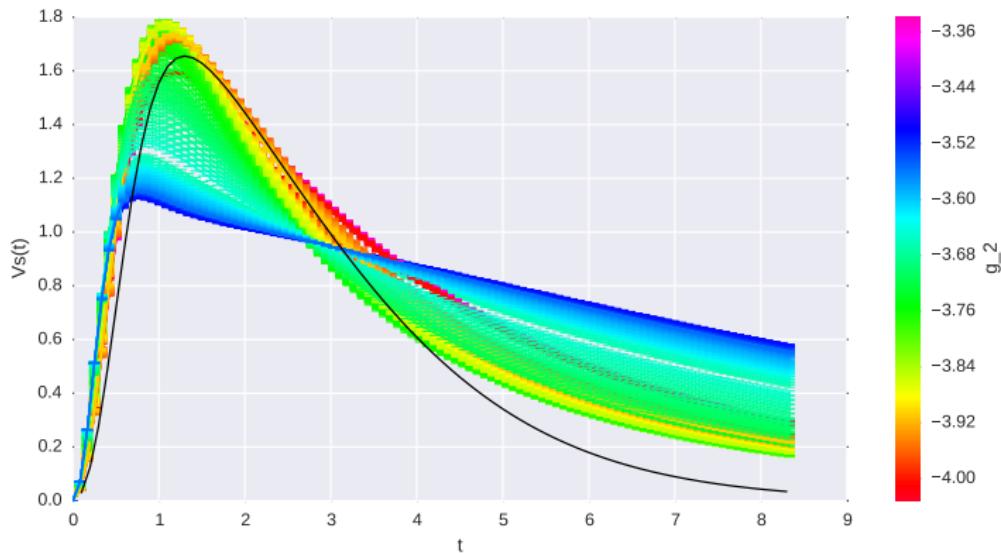
Summary

Results:
- Measure eigenvalues λ_i
- Interpretation $\Rightarrow$ spectral dimension
- Geometries change from 1d to 2d
- good agreement with fuzzy S^2

Future work:
- Scaling and critical exponents
- ζ-function
- construct more fuzzy spaces

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Thanks for listening!