Partially massless graviton on space-times beyond Einstein

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Outline

Fierz-Pauli theory on Einstein space-times

Einstein space-times

$$R_{\mu\nu} = \Lambda\, g_{\mu\nu}\,, \qquad R = 4\Lambda\,.$$

Linear equations of motion for a massive spin-2 field

$$\delta E_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} - \Lambda\Big(h_{\mu\nu} - \frac{1}{2} h\, g_{\mu\nu}\Big) + \frac{m^2}{2}\Big(h_{\mu\nu} - h\, g_{\mu\nu}\Big) = 0$$

with

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \equiv -\tfrac{1}{2}\left[\delta^\rho_\mu \delta^\sigma_\nu \nabla^2 + g^{\rho\sigma}\nabla_\mu\nabla_\nu - \delta^\rho_\mu \nabla^\sigma\nabla_\nu - \delta^\rho_\nu \nabla^\sigma \nabla_\mu - g_{\mu\nu}\eta^{\rho\sigma}\nabla^2 + g_{\mu\nu}\nabla^\rho\nabla^\sigma\right] h_{\rho\sigma}$$

- A massive graviton should propagate $2s+1=5$ degrees of freedom.
  - $\nabla^\nu \delta\bar{E}_{\mu\nu} \implies$ 4 vector constraints : $\nabla^\mu h_{\mu\nu} - \nabla_\nu h = 0$.
  - Taking another divergence : $\mathcal{C} \equiv 2\nabla^\mu\nabla^\nu \delta E_{\mu\nu} + m^2 g^{\mu\nu}\delta E_{\mu\nu} = \frac{m^2}{2}\Big(\Lambda - \frac{3m^2}{2}\Big)h = 0$.
  - Scalar constraint $h=0$ in general.

Partial masslessness on Einstein space-time

$$\mathcal{C} \equiv 2\left(\nabla^\mu \nabla^\nu + \frac{m^2}{2} g^{\mu\nu}\right) \delta E_{\mu\nu} = \frac{m^2}{2}\left(\Lambda - \frac{3m^2}{2}\right) h = 0$$

Partial masslessness

$$m^2 = \frac{2\Lambda}{3} \qquad \Longrightarrow \qquad \mathcal{C} = 0 \text{ identically.}$$

- The field equations $\delta E_{\mu\nu}$ and the quadratic action $S \sim \int \mathrm{d}^4 x \, h^{\mu\nu} \, \delta E_{\mu\nu}$ are invariant under an additional scalar gauge symmetry :

$$\Delta h_{\mu\nu} = \left(\nabla_\mu \nabla_\nu + \frac{m^2}{2} g_{\mu\nu}\right) \zeta(x)$$

- Four propagating degrees of freedom $\Longrightarrow$ **partially massless graviton.**

PARTIAL MASSLESSNESS ON EINSTEIN SPACE-TIME

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PARTIAL MASSLESSNESS

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HIGUCHI BOUND

- $m^2 > \frac{2\Lambda}{3}$ $\Longrightarrow$ 5 dof, stable,
- $m^2 = \frac{2\Lambda}{3}$ $\Longrightarrow$ 4 dof, stable PM graviton,
- $m^2 < \frac{2\Lambda}{3}$ $\Longrightarrow$ 5 dof, but the scalar one is the Higuchi ghost $\Rightarrow$ unstable.

Towards a non-linear theory for partial masslessness

Motivations

1. Old cosmological constant problem :
   - $\Lambda \sim m^2$ protected by a symmetry.
   - technically natural : $m^2 \sim 0$ restaure diffeomorphism invariance.
2. No scalar dof $\Longrightarrow$ no vDVZ discontinuity.
3. GR is a non-linear theory $\Longrightarrow$ non-linear massive spin-2 theory.

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The dRGT massive gravity theory

$$S = M_g^2 \int \mathrm{d}^4x\sqrt{|g|}\Big[R(g) - 2m^2 V\left(S;\beta_n\right)\Big], \qquad \text{with} \quad V(S;\beta_n) = \sum_{n=0}^{3} \beta_n e_n(S)$$

- ▷ Auxiliary metric $f_{\mu\nu}$, square-root matrix $S^\mu{}_\nu = \left[\sqrt{g^{-1}f}\right]^\mu{}_\nu$,
- ▷ $e_n(S)$ elementary symmetric polynomials :

$$e_0(S) = 1\,, \quad e_1(S) = \mathrm{Tr}[S]\,, \quad e_2(S) = \frac{1}{2}\Big(\mathrm{Tr}[S]^2 - \mathrm{Tr}[S^2]\Big)\,,$$
$$e_3(S) = \frac{1}{6}\Big(\mathrm{Tr}[S]^3 - 3\mathrm{Tr}[S]\mathrm{Tr}[S^2] + 2\mathrm{Tr}[S^3]\Big)\,, \quad e_4(S) = \det(S)\,.$$

Methodology

Our goal

Find a theory for a partially massless graviton propagating on (more) general backgrounds than Einstein's.

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Find a theory for a partially massless graviton propagating on (more) general backgrounds than Einstein's.

1. We start from the linearized massive gravity field equations around a general background space-time, obtained from the dRGT action with $\beta_3 = 0$,

$$\delta E_{\mu\nu} \equiv \delta \mathcal{G}_{\mu\nu} + m^2 \, \delta V_{\mu\nu} = 0 \, ,$$

2. We write the five covariant constraints and look for space-times for which the scalar constraint vanishes identically.
3. This is our candidate for partial masslessness and we determine the additional scalar gauge symmetry associated to it.

Our starting point

The background

- We consider the metric perturbation $\delta g_{\mu\nu}$ around an arbitrary background space-time $g_{\mu\nu}$.
- We define the matrix $S^\mu{}_\nu$ by the implicit background relation

$$R_{\mu\nu} = m^2 \left[\left(\beta_0 + \frac{1}{2}\beta_1 e_1 \right) g_{\mu\nu} + (\beta_1 + \beta_2 e_1)\, S_{\mu\nu} - \beta_2 [S^2]_{\mu\nu} \right]$$

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The linearized field equations

$$\delta E_{\mu\nu} \equiv \delta \mathcal{G}_{\mu\nu} + m^2 \delta V_{\mu\nu} = 0\,.$$

- No assumption on the unique and arbitrary background metric,

Our starting point

Linearized field equations

$$\delta E_{\mu\nu} \equiv \delta\mathcal{G}_{\mu\nu} + m^2 \delta V_{\mu\nu} = 0 \,.$$

where

$$\delta\mathcal{G}_{\mu\nu} = -\tfrac{1}{2}\Big[\delta^\rho_\mu \delta^\sigma_\nu \nabla^2 + g^{\rho\sigma}\nabla_\mu\nabla_\nu - \delta^\rho_\mu \nabla^\sigma\nabla_\nu - \delta^\rho_\nu\nabla^\sigma\nabla_\mu - g_{\mu\nu}g^{\rho\sigma}\nabla^2 + g_{\mu\nu}\nabla^\rho\nabla^\sigma + g_{\mu\nu}R^{\rho\sigma} - \delta^\rho_\mu\delta^\sigma_\nu R\Big]\delta g_{\rho\sigma} \,,$$

and

$$\begin{aligned}\delta V_{\mu\nu} = {}& (\beta_0 + \beta_1 e_1 + \beta_2 e_2)\,\delta g_{\mu\nu} - \frac{1}{2}(\beta_1 + \beta_2 e_1)\left(S^\sigma{}_\mu \delta g_{\nu\sigma} + S^\sigma{}_\nu \delta g_{\mu\sigma}\right) \\ & - \frac{1}{2}(\beta_1 + \beta_2 e_1)\, g_{\mu\nu} S^{\rho\sigma}\delta g_{\rho\sigma} + \frac{1}{2} S_{\mu\nu} S^{\rho\sigma}\delta g_{\rho\sigma} + \frac{1}{2} g_{\mu\nu}[S^2]^{\rho\sigma}\delta g_{\rho\sigma} \\ & - \frac{1}{2}(\beta_1 + \beta_2 e_1)\left(\delta S^\sigma{}_\mu g_{\nu\sigma} + \delta S^\sigma{}_\nu g_{\mu\sigma}\right) \,.\end{aligned}$$

with

$$\delta S = \frac{1}{2}\mathbb{X}^{-1}\sum_{k=1}^{4}\sum_{m=0}^{k-1}(-1)^m e_{4-k}(S) S^{k-m-2}\delta S^2 S^m \,,$$

The covariant constraints

The vector constraint

$$\nabla^{\nu}\delta E_{\mu\nu} = 0\,.$$

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The scalar constraint

1. We define the generalized traces and divergences of $\delta E_{\mu\nu}$ by :

$$\Phi_i \equiv [S^i]^{\mu\nu}\,\delta E_{\mu\nu}\,, \qquad 0 \leq i \leq 3\,,$$
$$\Psi_i \equiv [S^i]^{\mu\nu}\nabla_\nu \nabla^\lambda\,\delta E_{\lambda\mu}\,, \qquad 0 \leq i \leq 3\,.$$

2. The scalar constraint is

$$\mathcal{C} = \frac{m^2\,\beta_1\,e_4}{2}\,\Phi_0 + m^2\,\beta_2\,e_4\,\Phi_1 + e_3\,\Psi_0 - e_2\,\Psi_1 + e_1\,\Psi_2 - \Psi_3 = 0\,.$$

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The vector constraint

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Massive graviton propagating at most five degrees of freedom on an arbitrary background space-time.

Partially massless graviton on more general backgrounds

Our system of equations

- The background equation linking the Ricci tensor and the metric to the matrix S

$$R_{\mu\nu} = m^2 \left[\left(\beta_0 + \frac{1}{2}\beta_1 e_1 \right) g_{\mu\nu} + (\beta_1 + \beta_2 e_1)\, S_{\mu\nu} - \beta_2 [S^2]_{\mu\nu} \right] ,$$

- The scalar constraint

$$\mathcal{C} = m^2 \left(A^{\beta\lambda} + \tilde{A}^{\beta\lambda} \right) \delta g_{\beta\lambda} + B_\rho^{\beta\lambda} \nabla^\rho \delta g_{\beta\lambda} = 0 \,.$$

▷ $\tilde{A}^{\beta\lambda}$ and $B_\rho^{\beta\lambda}$ are complicated expressions containing only derivatives of S : $\nabla_\rho S_{\mu\nu}$, $\nabla_\sigma \nabla_\rho S_{\mu\nu}$,

▷ $A^{\beta\lambda}$ depends only on S :

$$A^{\beta\lambda} = m^2 \Big[\left(\beta_0\beta_1 + \beta_0\beta_2 e_1 + \frac{1}{2}\beta_1^2 e_1 \right) g^{\beta\lambda} + \left(-2\beta_0\beta_2 - \frac{1}{2}\beta_1^2 - 2\beta_2^2 e_2 + \beta_2^2 e_1^2 \right) S^{\beta\lambda} - \beta_2^2 e_1 [S^2]^{\beta\lambda} \Big]$$

Simplifying the analysis

▷ We now restrict to the case $\nabla_\rho S_{\mu\nu} = 0$, which implies $\nabla_\rho R_{\mu\nu} = 0$.

Ricci symmetric space-times : $\nabla_\rho R_{\mu\nu} = 0$

- of Petrov type N : restricted pp-waves with $R = 0$,
- of Petrov type O : vanishing Weyl tensor, conformally flat.
- of Petrov type D : 2+2 decomposable.

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$$R_{\mu\nu} = m^2 \left[\left(\beta_0 + \frac{1}{2}\beta_1 e_1 \right) g_{\mu\nu} + \left(\beta_1 + \beta_2 e_1 \right) S_{\mu\nu} - \beta_2 [S^2]_{\mu\nu} \right] ,$$

$$\mathcal{C} = m^2 A^{\beta\lambda} \delta g_{\beta\lambda} = 0 \, .$$

$$A^{\beta\lambda} = m^2 \left[\left(\beta_0\beta_1 + \beta_0\beta_2 e_1 + \frac{1}{2}\beta_1^2 e_1 \right) g^{\beta\lambda} + \left(-2\beta_0\beta_2 - \frac{1}{2}\beta_1^2 - 2\beta_2^2 e_2 + \beta_2^2 e_1^2 \right) S^{\beta\lambda} - \beta_2^2 e_1 [S^2]^{\beta\lambda} \right]$$

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OUR EQUATIONS

$$R_{\mu\nu} = m^2 \left[\left(\beta_0 + \frac{1}{2}\beta_1 e_1 \right) g_{\mu\nu} + (\beta_1 + \beta_2 e_1)\, S_{\mu\nu} - \beta_2 [S^2]_{\mu\nu} \right] ,$$

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▷ **Several solutions of Petrov type D and O more general than Einstein space-times.**

Constructive approach

- Ricci symmetric space-times : $\nabla_\rho R_{\mu\nu} = 0 \Longrightarrow [R^2]_{\mu\nu} = r_1 g_{\mu\nu} + r_2 R_{\mu\nu}$.

More general field equations

$$\delta E_{\mu\nu} = \delta \mathcal{G}_{\mu\nu} + a_1\, \delta g_{\mu\nu} + a_2\, g_{\mu\nu} g^{\rho\sigma} \delta g_{\rho\sigma} + b_1\, \left(R^\rho_\mu \delta g_{\nu\rho} + R^\rho_\nu \delta g_{\mu\rho} \right) + b_2\, g_{\mu\nu} R^{\rho\sigma} \delta g_{\rho\sigma}$$
$$+ b_3\, R_{\mu\nu} g^{\rho\sigma} \delta g_{\rho\sigma} + b_4\, R_{\mu\nu} R^{\rho\sigma} \delta g_{\rho\sigma} + b_5\, R^\rho_\mu R^\sigma_\nu \delta g_{\rho\sigma}$$

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$$+ b_3\,R_{\mu\nu} g^{\rho\sigma}\delta g_{\rho\sigma} + b_4\,R_{\mu\nu} R^{\rho\sigma}\delta g_{\rho\sigma} + b_5\,R^\rho_\mu R^\sigma_\nu \delta g_{\rho\sigma}$$

More general scalar constraint

$$\mathcal{C} = c_0\,\nabla^\nu\nabla^\mu\delta E_{\mu\nu} + c_1\,R^\nu_\rho \nabla^\rho\nabla^\nu\delta E_{\mu\nu} + c_2\,g^{\mu\nu}\delta E_{\mu\nu} + c_3\,R^{\mu\nu}\delta E_{\mu\nu} = 0$$

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We then restrict the parameter space by imposing that

1. the action $S = M_{\rm g}\int \mathrm{d}^4x\sqrt{-g}\delta g^{\mu\nu}\delta E_{\mu\nu}$ is symmetric (i.e. give back $\delta E_{\mu\nu} = 0$),
2. the scalar constraint is indeed a constraint and vanishes identically.

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- ▷ Ricci symmetric space-times : $\nabla_\rho R_{\mu\nu} = 0 \Longrightarrow [R^2]_{\mu\nu} = r_1 g_{\mu\nu} + r_2 R_{\mu\nu}$.

More general field equations

$$\delta E_{\mu\nu} = \delta\mathcal{G}_{\mu\nu} + a_1\,\delta g_{\mu\nu} + a_2\,g_{\mu\nu}g^{\rho\sigma}\delta g_{\rho\sigma} + b_1\,\left(R^\rho_\mu\delta g_{\nu\rho} + R^\rho_\nu\delta g_{\mu\rho}\right) + b_2\,g_{\mu\nu}R^{\rho\sigma}\delta g_{\rho\sigma} + b_3\,R_{\mu\nu}g^{\rho\sigma}\delta g_{\rho\sigma} + b_4\,R_{\mu\nu}R^{\rho\sigma}\delta g_{\rho\sigma} + b_5\,R^\rho_\mu R^\sigma_\nu\delta g_{\rho\sigma}$$

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2. the scalar constraint is indeed a constraint and vanishes identically.

- ▷ **There exists partially massless solutions of Petrov-type N, O and D for more general equations than the ones coming from dRGT massive gravity.**

Summary & Prospects

Partial masslessness

- Starting from the linearized field equations we study partial masslessness by studying the Lagrangian constraints.
- We have found several backgrounds more general than Einstein for which the scalar constraints identically vanishes. These are backgrounds on which a partially massless graviton propagates.
- Reverse constructive approach $\Longrightarrow$ Partial masslessness for more general field equations than the ones coming from the dRGT action.

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Prospects

- Perform a general background analysis (*i.e.* not restricted to Ricci symmetric background) ?
- Does there exist a fully non-linear partially massless gravity theory ?
- Is dRGT the more general massive gravity theory ?