

Hawking radiation from magnetized black holes

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$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (1)$$

$$r_s = 2M \quad T_H = \kappa/2\pi \quad T_H = 1/8\pi M \quad (2)$$

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (3)$$

The electromagnetic four potential is given by

$$A^\alpha = \left(\frac{q}{r}, 0, 0, 0\right) \quad (4)$$

The event horizons are

$$r_{\pm} = M \pm \sqrt{M^2 - q^2} \quad (5)$$

Kerr-Newman black hole

The Kerr-Newman geometry in Boyer and Lindquist coordinates has the form

$$ds^2 = -\frac{\Delta}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)d\phi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \quad (6)$$

where

$$\begin{aligned} \Delta &= r^2 - 2Mr + q^2 + a^2, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta, \\ a &= \frac{J}{M}, \end{aligned}$$

where J is the angular momentum of the black hole. The electromagnetic vector potential can be written as

$$A = (\Phi_0 - \omega \Phi_3) dt + \Phi_3 d\phi,$$

with

$$\Phi_0 = -\frac{qr(r^2 + a^2)}{\rho^2}, \quad \Phi_3 = \frac{aqr \sin^2 \theta}{\rho^2}.$$

This black hole has two horizons given by

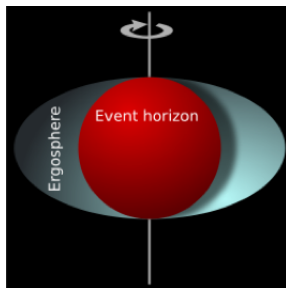
$$r^2 - 2Mr + q^2 + a^2 = 0.$$

Its solutions are

$$r_{\pm} = M \pm \sqrt{M^2 - q^2 - a^2}, \quad (7)$$

which are the expressions for the location of inner and outer horizons. Hawking temperature is

$$T_H = \frac{\sqrt{M^2 - a^2 - q^2}}{2\pi \left(a^2 + (M + \sqrt{M^2 - a^2 - q^2})^2 \right)}. \quad (8)$$



The one-parameter static solution of the coupled Einstein–Maxwell system is given by the metric

$$ds^2 = \left(1 + \frac{1}{4}B^2\rho^2\right)^2(-dt^2 + d\rho^2 + dz^2) + \left(1 + \frac{1}{4}B^2\rho^2\right)^{-2}\rho^2 d\phi^2 \quad (9)$$

with $t, z \in (-\infty, +\infty)$, $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$. The electromagnetic field can be described by the Maxwell tensor

$$F = e^{-i\psi} B(dz \wedge dt) \quad (10)$$

where ψ is real parameter of duality rotation. In particular, for $\psi = 0$, the Maxwell tensor is $F = Bdz \wedge dt$ which describes an electric field pointing along the z -direction, whereas for $\psi = \pi/2$ one obtains $F = B(1 + 1/4B^2\rho^2)^{-2}\rho d\rho \wedge d\phi$, which represents a purely magnetic field oriented along the z -direction.

Harrison's transformation and Ernst's technique

In 1976, F. J. Ernst using Harrison transformation presented a procedure which was used for transforming asymptotically flat axially symmetric solutions of the coupled Einstein-Maxwell equations into solution resembling Melvin's magnetic universe. He used this technique for the removal of the nodal singularity of the C -metric and studied the Schwarzschild, Reissner-Nordström and Kerr black holes in Melvin universe.

Schwarzschild black hole in Melvin universe

The Harrison transformations can be used to magnetize the Schwarzschild black hole. The line element becomes

$$ds^2 = (1 + \frac{1}{4}B^2 r^2 \sin^2 \theta)^2 [-(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\theta^2] \quad (11)$$

$$+ (1 + \frac{1}{4}B^2 r^2 \sin^2 \theta)^{-2} r^2 \sin^2 \theta d\phi^2 \quad (12)$$

In this case the magnetic field components are given by

$$H_r = \Lambda^{-2} B \cos \theta \quad (13)$$

$$H_\theta = -\Lambda^{-2} B (1 - \frac{2M}{r})^{1/2} \sin \theta \quad (14)$$

$$\Lambda = 1 + B\Phi - \frac{1}{4}B^2 \varepsilon = 1 + \frac{1}{4}B^2 r^2 \sin^2 \theta, \quad (15)$$
$$(16)$$

Note that if $M = 0$ the above metric becomes the Melvin's magnetic universe, while for $M \neq 0$ there is an event horizon at $r = 2M$ and the angular component of magnetic field vanishes at the event horizon. Further, the metric has singularity at $r = 0$, as in case of Schwarzschild metric.

Reissner-Nordström black hole in Melvin universe

The Reissner-Nordström black hole in Melvin universe is

$$ds^2 = (\Lambda)^2 \left[-\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 \right] \quad (17)$$

$$+ (\Lambda)^{-2} r^2 \sin^2 \theta (d\phi - \omega' dt)^2 \quad (18)$$

where

$$\Lambda = 1 + B\Phi - \frac{1}{4}B^2 = 1 + \frac{1}{4}B^2(r^2 \sin^2 \theta + q^2 \cos^2 \theta) - iBq \cos \theta, \quad (19)$$

$$(20)$$

If $q = 0$ then this metric reduces to the Schwarzschild black hole in Melvin universe, and if $B = 0$, then this becomes Reissner-Nordström.

The components of the electric and magnetic fields from the electromagnetic potential Φ are

$$H_r + iE_r = \Lambda^{-2} \left[i \left(\frac{e}{r^2} \right) \left\{ 1 - \frac{1}{4} B^2 (r^2 \sin^2 \theta + q^2 \cos^2 \theta) \right\} \right. \quad (21)$$

$$\left. + B \left(1 - \frac{1}{2} i B q \cos \theta \right) \left(1 - \frac{q^2}{r^2} \right) \cos \theta \right], \quad (22)$$

and

$$H_\theta + iE_\theta = -B |\Lambda|^2 \left(1 - \frac{1}{2} i q^2 \cos \theta \right) \left(1 - \frac{2M}{r} + \frac{q^2}{r^2} \right)^{1/2} \sin \theta.$$

Magnetized Kerr-Newman black hole

The magnetized Kerr-Newman black hole of mass M , angular momentum per unit mass a , carrying an electric charge q , embedded in a uniform background magnetic field B is

$$ds^2 = H[-f dt^2 + R^2(\frac{dr^2}{\Delta} + d\theta^2)] + \frac{\Sigma \sin^2 \theta}{HR^2}(d\phi - \omega dt)^2, \quad (23)$$

where

$$R^2 = r^2 + a^2 \cos^2 \theta, \quad (24)$$

$$\Delta = (r^2 + a^2) - 2Mr + q^2, \quad (25)$$

$$\Sigma = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad (26)$$

$$f = \frac{R^2 \Delta}{\Sigma}, \quad (27)$$

$$H = 1 + \frac{H_{(1)}B + H_{(2)}B^2 + H_{(3)}B^3 + H_{(4)}B^4}{R^2}, \quad (28)$$

with

$$\begin{aligned}
H_{(1)} &= 2aqr \sin^2 \theta - 2p(r^2 + a^2) \cos \theta, \\
H_{(2)} &= \frac{1}{2}[(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] \sin^2 \theta + \frac{3}{2}q^2(a^2 + r^2 \cos^2 \theta), \\
H_{(3)} &= -\frac{qa\Delta}{2r}[r^2(3 - \cos^2 \theta) \cos^2 \theta + a^2(1 + \cos^2 \theta)] + \frac{1}{2}p(r^4 - a^4) \sin^2 \theta \cos \theta \\
&\quad + \frac{q\bar{q}a[(2r^2 + a^2) \cos^2 \theta + a^2]}{2r} - pa^2 \Delta \sin^2 \theta \cos \theta - \frac{1}{2}p\bar{q}^2(r^2 + a^2) \cos^3 \theta \\
&\quad + \frac{aq(r^2 + a^2)^2(1 + \cos^2 \theta)}{2r}, \\
H_{(4)} &= \frac{1}{16}(r^2 + a^2)^2 R^2 \sin^4 \theta + \frac{1}{4}M^2 a^2[r^2(\cos^2 \theta - 3)^2 \cos^2 \theta + a^2(1 + \cos^2 \theta)^2] \\
&\quad + \frac{1}{4}Ma^2 r(r^2 + a^2) \sin^6 \theta + \frac{1}{16}\bar{q}^4[r^2 \cos^2 \theta + a^2(1 + \sin^2 \theta)] \cos^2 \theta \\
&\quad + \frac{1}{4}Ma^2 r(r^2 + a^2) \sin^6 \theta + \frac{1}{4}Ma^2 \bar{q}^2 r(\cos^2 \theta - 5) \sin^2 \theta \cos^2 \theta \\
&\quad + \frac{1}{8}\bar{q}^2(r^2 + a^2)(r^2 + a^2 + a^2 \sin^2 \theta) \sin^2 \theta \cos^2 \theta.
\end{aligned}$$

Here

$$\bar{q}^2 = q^2 + p^2,$$

and

$$\omega = \frac{1}{\Sigma} [(2Mr - \bar{q}^2)a + \omega_{(1)}B + \omega_{(2)}B^2 + \omega_{(3)}B^3 + \omega_{(4)}B^4], \quad (29)$$

with

$$\omega_{(1)} = -2qr(r^2 + a^2) + 2ap\Delta \cos \theta,$$

$$\omega_{(2)} = -\frac{3}{2}a\bar{q}^2(r^2 + a^2 + \Delta \cos^2 \theta),$$

$$\begin{aligned} \omega_{(3)} = & 4qM^2a^2r + \frac{1}{2}ap\bar{q}^4 \cos^3 \theta + \frac{1}{2}qr(r^2 + a^2)[r^2 - a^2 + (r^2 + 3a^2) \cos^2 \theta] \\ & + \frac{1}{2}ap(r^2 + a^2)[3r^2 + a^2 - (r^2 - a^2) \cos^2 \theta] \cos \theta - aM\bar{q}^2(2aq + pr \cos^3 \theta) \\ & - apMr[2R^2 + (r^2 + a^2) \sin^2 \theta] \cos \theta + \frac{1}{2}ap\bar{q}^2[3r^2 + a^2 + 2a^2 \cos^2 \theta] \cos \theta \\ & + \frac{1}{2}q\bar{q}^2r[(r^2 + 3a^2) \cos^2 \theta - 2a^2] + qM[r^4 - a^4 + r^2(r^2 + 3a^2) \sin^2 \theta], \end{aligned}$$

$$\begin{aligned} \omega_{(4)} = & \frac{1}{2}a^3M^3r(3 + \cos^4 \theta) - \frac{1}{8}a\bar{q}^4[r^2(2 + \sin^2 \theta) \cos^2 \theta + a^2(1 + \cos^4 \theta)] \\ & + \frac{1}{16}a\bar{q}^2(r^2 + a^2)[r^2(1 - 6 \cos^2 \theta + 3 \cos^4 \theta) - a^2(a + \cos^4 \theta)] \\ & - \frac{1}{4}a^3M^2\bar{q}^2(3 + \cos^4 \theta) - \frac{1}{16}a\bar{q}^6 \cos^4 \theta + \frac{1}{4}aM^2[r^4(3 - 6 \cos^2 \theta + 3 \cos^4 \theta) \\ & + 2a^2r^2(3 \sin^2 \theta - 2 \cos^4 \theta) - a^4(1 + \cos^4 \theta)] + \frac{1}{2}aM\bar{q}^4r \cos^4 \theta \end{aligned}$$

The electromagnetic vector potential is

$$A = (\Phi_0 - \omega\Phi_3)dt + \Phi_3 d\phi, \quad (30)$$

where

$$\Phi_0 = \frac{\Phi_0^{(0)} + \Phi_0^{(1)}B + \Phi_0^{(2)}B^2 + \Phi_0^{(3)}B^3}{4\Sigma}, \quad (31)$$

with

$$\begin{aligned} \Phi_0^{(0)} &= 4[-qr(r^2 + a^2) + ap\Delta \cos \theta], \\ \Phi_0^{(1)} &= -6a\bar{q}^2(r^2 + a^2 + \Delta \cos^2 \theta), \\ \Phi_0^{(2)} &= -3q[(r + 2M)a^4 - (r^2 + 4Mr + \Delta \cos^2 \theta)r^3 + a^2(2\bar{q}^2(r + 2M) - 6Mr^2 \\ &\quad - 8M^2r - 3r\Delta \cos^2 \theta)] + 3p\Delta[3ar^2 + a^3 + a(a^2 + \bar{q}^2 - r^2) \cos^2 \theta] \cos \theta, \\ \Phi_0^{(3)} &= -\frac{1}{2}a[4a^4M^2 + 12a^2M^2\bar{q}^2 + 2a^2\bar{q}^4 + 2a^4Mr - 24a^2M^3r + 4a^2M\bar{q}^2r \\ &\quad - 24a^2M^2r^2 - 4a^2Mr^3 - \bar{q}^2r^4 - 6Mr^5 - 6r\Delta\{2M(r^2 + a^2) - \bar{q}^2r\} \cos^2 \theta \\ &\quad + a^4\bar{q}^2 - 12M^2r^4 + \Delta(\bar{q}^4 - 3\bar{q}^2r^2 + a^2(4M^2 + \bar{q}^2 - 6Mr)) \cos^4 \theta], \end{aligned}$$

and

$$\Phi_3 = \frac{\Phi_3^{(0)} + \Phi_3^{(1)} + \Phi_3^{(2)} + \Phi_3^{(3)}}{R^2 H}, \quad (32)$$

with

$$\begin{aligned} \Phi_3^{(0)} &= aqr \sin^2 \theta - p(r^2 + a^2) \cos \theta, \\ \Phi_3^{(1)} &= \frac{1}{2} [\Sigma \sin^2 \theta + 3\bar{q}^2 (a^2 + r^2 \cos^2 \theta)], \\ \Phi_3^{(2)} &= \frac{3}{4} aqr(r^2 + a^2) \sin^4 \theta - \frac{3}{4} p(r^2 + a^2)^2 \sin^2 \theta \cos \theta + 3a^2 pMr \sin^2 \theta \cos \theta \\ &\quad + \frac{3}{2} aqm[r^2(3 - \cos^2 \theta) \cos^2 \theta + a^2(1 + \cos^2 \theta)] - \frac{3}{4} aq\bar{q}^2 r \sin^2 \theta \cos^2 \theta \\ &\quad - \frac{3}{4} p\bar{q}^2 [(r^2 - a^2) \cos^2 \theta + 2a^2] \cos \theta, \\ \Phi_3^{(3)} &= \frac{1}{4} \bar{q}^2 (r^2 + a^2) [r^2 + a^2 + a^2 \sin^2 \theta \cos^2 \theta] - \frac{1}{2} a^2 \bar{q}^2 Mr (5 - \cos^2 \theta) \sin^2 \theta \cos^2 \theta \\ &\quad + \frac{1}{2} a^2 M^2 [r^2 (3 - \cos^2 \theta)^2 \cos^2 \theta + a^2 (1 + \cos^2 \theta)^2] + \frac{1}{2} a^2 Mr (r^2 + a^2) \sin^6 \theta \\ &\quad + \frac{1}{8} R^2 (r^2 + a^2)^2 \sin^4 \theta + \frac{1}{8} \bar{q}^4 [r^2 \cos^2 \theta + a^2 (2 - \cos^2 \theta)^2] \cos^2 \theta. \end{aligned}$$

Quantum tunneling of charged scalar particles

We discuss quantum tunneling of charged scalar particles from event horizon of the magnetized Kerr-Newman black hole using Hamilton Jacobi method which is a semi-classical approach.

First we write the metric in the form

$$ds^2 = -\left(fH - \frac{\Sigma \sin^2 \theta \omega^2}{HR^2}\right) dt^2 + \frac{HR^2}{\Delta} dr^2 + HR^2 d\theta^2 + \frac{\Sigma \sin^2 \theta}{HR^2} d\phi^2 - 2 \frac{\Sigma \sin^2 \theta \omega}{HR^2} dt d\phi \quad (33)$$

Using new functions $C(r, \theta)$, $g(r, \theta)$, $D(r, \theta)$, $K(r, \theta)$ and $G(r, \theta)$ this becomes

$$ds^2 = -C(r, \theta) dt^2 + \frac{dr^2}{g(r, \theta)} + D(r, \theta) d\theta^2 + K(r, \theta) d\phi^2 - 2G(r, \theta) dt d\phi, \quad (34)$$

$$C(r, \theta) = fH - \frac{\Sigma \sin^2 \theta \omega^2}{HR^2},$$

$$g(r, \theta) = \frac{\Delta}{HR^2},$$

$$D(r, \theta) = HR^2,$$

$$K(r, \theta) = \frac{\Sigma \sin^2 \theta}{HR^2},$$

$$G(r, \theta) = \frac{\Sigma \sin^2 \theta \omega}{HR^2}.$$

The event horizons of the magnetized Kerr-Newman black hole is obtained from

$$(g_{rr})^{-1} = g(r, \theta) = \frac{\Delta}{HR^2}.$$

The outer and inner horizons corresponding to this black hole denoted by " r_{\pm} " are given by

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - q^2 - p^2}. \quad (35)$$

The angular velocity at black hole horizon is

$$\Omega_H = \frac{G(r_+, \theta)}{K(r_+, \theta)}. \quad (36)$$

Using the values of $G(r_+, \theta)$ and $K(r_+, \theta)$ this becomes

$$\Omega_H = \omega(r_+, \theta). \quad (37)$$

We define new function

$$F(r, \theta) = -(g^{tt})^{-1} = C(r, \theta) + \frac{G^2(r, \theta)}{K(r, \theta)}. \quad (38)$$

Using the value of $C(r, \theta)$, $G(r, \theta)$ and $K(r, \theta)$ we will get

$$F(r, \theta) = Hf. \quad (39)$$

Now we shall solve the Klein-Gordon equation to study the quantum tunneling of charged scalar particles from event horizons of magnetized Kerr-Newman black hole. The Klein-Gordon equation for scalar field Φ is

$$g^{\alpha\beta}(\partial_\alpha - \frac{ie_s}{\hbar}A_\alpha)(\partial_\beta - \frac{ie_s}{\hbar}A_\beta)\Phi - \frac{m_s^2}{\hbar^2}\Phi = 0, \quad (40)$$

where $\alpha, \beta = 1, 2, 3, 4$ corresponds to t, r, θ, ϕ respectively, e_s is the charge, m_s is its mass, $g^{\alpha\beta}$ is the inverse of the metric tensor and A_α is the vector potential which is given by Eq. (30).

Applying the Wentzel-Kramen-Brillouin (WKB) approximation and assuming the following ansatz

$$\Phi(t, r, \theta, \phi) = \exp\left[\frac{i}{\hbar} I(t, r, \theta, \phi) + I_1(t, r, \theta, \phi) + O(\hbar)\right], \quad (41)$$

where I is the action.

After simplifying we get

$$\begin{aligned}
 (\partial_t - \frac{ie_s}{\hbar} A_t)^2 \Phi &= \exp(\frac{i}{\hbar} I + I_1) [(\frac{i}{\hbar} \partial_t I + \partial_t I_1)^2 + (\frac{i}{\hbar} \partial_{tt} I + \partial_{tt} I_1) - \frac{ie_s}{\hbar} \partial_t A_t \\
 &\quad - \frac{ie_s}{\hbar} A_t (\frac{i}{\hbar} \partial_t I + \partial_t I_1) - \frac{ie_s}{\hbar} A_t (\frac{i}{\hbar} \partial_t I + \partial_t I_1) + (\frac{ie_s A_t}{\hbar})^2]. \quad (42)
 \end{aligned}$$

$$\begin{aligned}
 (\partial_r - \frac{ie_s}{\hbar} A_r)^2 \Phi &= \exp(\frac{i}{\hbar} I + I_1) [(\frac{i}{\hbar} \partial_r I + \partial_r I_1)^2 + (\frac{i}{\hbar} \partial_{rr} I + \partial_{rr} I_1) - \frac{ie_s}{\hbar} \partial_r A_r \\
 &\quad - \frac{ie_s}{\hbar} A_r (\frac{i}{\hbar} \partial_r I + \partial_r I_1) - \frac{ie_s}{\hbar} A_r (\frac{i}{\hbar} \partial_r I + \partial_r I_1) + (\frac{ie_s A_r}{\hbar})^2], \quad (43)
 \end{aligned}$$

$$\begin{aligned}
 (\partial_\theta - \frac{ie_s}{\hbar} A_\theta)^2 \Phi &= \exp(\frac{i}{\hbar} I + I_1) [(\frac{i}{\hbar} \partial_\theta I + \partial_\theta I_1)^2 + (\frac{i}{\hbar} \partial_{\theta\theta} I + \partial_{\theta\theta} I_1) - \frac{ie_s}{\hbar} \partial_\theta A_\theta \\
 &\quad - \frac{ie_s}{\hbar} A_\theta (\frac{i}{\hbar} \partial_\theta I + \partial_\theta I_1) - \frac{ie_s}{\hbar} A_\theta (\frac{i}{\hbar} \partial_\theta I + \partial_\theta I_1) + (\frac{ie_s A_\theta}{\hbar})^2], \quad (44)
 \end{aligned}$$

$$\begin{aligned}
(\partial_\phi - \frac{ie_s}{\hbar} A_\phi)^2 \Phi &= \exp(\frac{i}{\hbar} I + I_1) [(\frac{i}{\hbar} \partial_t I + \partial_t I_1)^2 + (\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) - \frac{ie_s}{\hbar} \partial_t A_\phi \\
&\quad - \frac{ie_s}{\hbar} A_\phi (\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) - \frac{ie_s}{\hbar} A_t (\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) + (\frac{ie_s A_\phi}{\hbar})^2], \quad (45)
\end{aligned}$$

and

$$\begin{aligned}
(\partial_t - \frac{ie_s}{\hbar} A_t)(\partial_\phi - \frac{ie_s}{\hbar} A_\phi) \Phi &= \exp(\frac{i}{\hbar} I + I_1) [(\frac{i}{\hbar} \partial_t I + \partial_t I_1)(\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) \\
&\quad + (\frac{i}{\hbar} \partial_{t\phi} I + \partial_{t\phi} I_1) - \frac{ie_s}{\hbar} A_\phi (\frac{i}{\hbar} \partial_t I + \partial_t I_1) \\
&\quad - \frac{ie_s}{\hbar} \partial_t A_\phi - \frac{ie_s}{\hbar} A_t (\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) + (\frac{ie_s}{\hbar})^2 A_t A_\phi]. \quad (46)
\end{aligned}$$

The above equation can also written as

$$g^{\alpha\beta}(\partial_\alpha I - e_s A_\alpha)(\partial_\beta I - e_s A_\beta) + m_s^2 = 0, \quad (47)$$

which after using the values of $g^{\alpha\beta}$ where $\alpha, \beta = 1, 2, 3, 4$, and simplification reduces to the following equation

$$\begin{aligned} g(r, \theta)(\partial_r I)^2 - \frac{2H(r, \theta)}{F(r, \theta)K(r, \theta)}(\partial_t I - e_s A_t)(\partial_\phi I - e_s A_\phi) - \frac{(\partial_t I - e_s A_t)^2}{F(r, \theta)} \\ - \frac{(\partial_t I - e_s A_t)^2}{F(r, \theta)} + \frac{C(r, \theta)}{F(r, \theta)K(r, \theta)}(\partial_\phi I - e_s A_\phi)^2 + \frac{(\partial_\theta I)^2}{(D(r, \theta))^2} + m_s^2 = 0. \end{aligned} \quad (48)$$

For the calculation of tunneling probability consider an ansatz of the form

$$I = -tE_s + \phi J_s + W(r, \theta), \quad (49)$$

where E_s and J_s are energy and angular momentum of scalar particles.

For fixed value of $\theta = \theta_0 = 0$, we have

$$I = -tE_s + \phi J_s + S(r) + \Theta(\theta_0). \quad (50)$$

Using these in Eq. (48) we have

$$\begin{aligned} & \frac{2H(r, \theta_0)}{F(r, \theta_0)K(r, \theta_0)}(E_s + e_s A_t)(J_s - e_s A_\phi) + \frac{C(r, \theta_0)}{F(r, \theta_0)K(r, \theta_0)}(J_s - e_s A_\phi)^2 \\ & + g(r, \theta_0)(S'(r))^2 - \frac{(E_s + e_s A_t)^2}{F(r, \theta_0)} + m_s^2 = 0. \end{aligned} \quad (51)$$

Using Taylor's theorem we obtain

$$F(r, \theta_0) = (r - r_+) F_r(r_+, \theta_0) = \frac{2(r - r_+)(r_+ - M)H(r_+)R^2(r_+)}{\Sigma(r_+, \theta_0)}, \quad (52)$$

$$g(r, \theta_0) = (r - r_+) g_r(r_+, \theta_0) = \frac{2(r - r_+)(r_+ - M)}{H(r_+)(r_+^2 + a^2 \cos^2 \theta_0)}, \quad (53)$$

$$\Omega_H = \frac{G(r_+, \theta_0)}{K(r_+, \theta_0)} = \omega(r_+, \theta_0). \quad (54)$$

Since the above equation is quadratic in terms of $S(r)$, so we have two solutions, one solution corresponds to scalar particles moving away from the black hole and the other solution corresponds to particles moving towards the black hole. Thus

$$S_{\pm}(r) = \pm \int \frac{(r_+^2 + a^2)dr}{2(r_+ - M)(r - r_+)} \times \sqrt{(E_s - \Omega_H J_s - e_s \Phi_o)^2 - \frac{2H(r_+)(r_+ - M)(r - r_+)(r_+^2 + a^2 \cos^2 \theta_0)}{(r_+^2 + a^2)^2} \left[\frac{(J_s - e_s A_\phi)^2}{K(r_+, \theta_0)} + m_s^2 \right]}. \quad (55)$$

Integrating the above equation using residue theory we will get

$$ImS_+ = \frac{\pi}{2} \frac{(E_s - \Omega_H J_s - e_s \Phi_o(r_+, \theta_0))(r_+^2 + a^2)}{(r_+ - M)}, \quad (56)$$

and

$$ImS_- = -\frac{\pi}{2} \frac{(E_s - \Omega_H J_s - e_s \Phi_o(r_+, \theta_0))(r_+^2 + a^2)}{(r_+ - M)}. \quad (57)$$

We note that

$$ImS_+ = -ImS_-. \quad (58)$$

The tunneling probabilities of particles in each direction are

$$P_{out} = \exp[-2ImI] = \exp[-2(ImS_+ + Im\Theta)], \quad (59)$$

$$P_{in} = \exp[-2ImI] = \exp[-2(ImS_- + Im\Theta)]. \quad (60)$$

Thus the resulting tunneling probability $\Gamma_{(mag)}$ of a particle from inside to outside the horizon of magnetized Kerr-Newman black hole is

$$\Gamma_{(mag)} \propto \frac{P_{out}}{P_{in}} = \frac{\exp[-2(ImS_+ + Im\Theta)]}{\exp[-2(ImS_- + Im\Theta)]},$$

or

$$\Gamma_{(mag)} = \exp[-2(ImS_+ - ImS_-)]. \quad (61)$$

$$\Gamma_{(mag)} = \exp[-4ImS_+], \quad (62)$$

which after using Eq. (56) the tunneling probability of scalar particle becomes

$$\Gamma_{(mag)} = \exp \left[\frac{-2\pi (r_+^2 + a^2)}{(r_+ - M)} (E_s - \Omega_H J_s - e_s \Phi_o(r_+, \theta_0)) \right]. \quad (63)$$

Note that the tunneling probability depends upon the mass M , angular momentum per unit mass a , the electric charge q of the black hole, as well as charge e_s , the energy E_s , and the angular momentum J_s of the scalar particles and the background magnetic field B .

In the absence of magnetic field i.e. for $B = 0$

The tunneling probability of scalar particle from the event horizon of magnetized Kerr-Newman black hole given by Eq. (35), in the absence of magnetic field i.e. for $B = 0$, should give us the tunneling probability of Kerr-Newman black hole. Under this condition Eqs. (31) imply

$$\Phi_o(r_+, \theta_0) = \frac{\Phi_0^{(0)}(r_+, \theta_0)}{4\Sigma(r_+, \theta_0)}, \quad (64)$$

or

$$\Phi_o(r_+, \theta_0) = \frac{-4qr_+(r_+^2 + a^2)}{4(r_+^2 + a^2)^2} = \frac{-qr_+}{(r_+^2 + a^2)}, \quad (65)$$

so that Eqs. (35) yields

$$\Gamma = \exp \left[\frac{-2\pi (r_+^2 + a^2)}{(r_+ - M)} \left(E_s - \Omega_H J_s - \frac{e_s qr_+}{(r_+^2 + a^2)} \right) \right], \quad (66)$$

which is the tunneling probability of Kerr-Newman black hole as given in the literature [?].

Weak magnetic field

If we consider the situation where the magnetic field is weak i.e. $B^n = 0$ for $n \geq 2$, Eq. (31) becomes

$$\Phi_0(r_+, \theta_0) = \frac{\Phi_0^{(0)}(r_+, \theta_0) + B\Phi_0^{(1)}(r_+, \theta_0)}{4\Sigma(r_+, \theta_0)}. \quad (67)$$

Or

$$\Phi_0(r_+, \theta_0) = -\frac{(4qr_+ + 6Ba\bar{q}^2)}{4(r_+^2 + a^2)}. \quad (68)$$

So the tunneling probability in the weak magnetic field is obtained as

$$\Gamma_{(mag)} = \exp \left[\frac{-2\pi (r_+^2 + a^2)}{(r_+ - M)} \left(E_s - \Omega_H J_s - \frac{e_s q r_+}{(r_+^2 + a^2)} \right) \right] \exp \left(\frac{3\pi B a \bar{q}^2}{(r_+ - M)} \right). \quad (69)$$

Using Eq. (66) we obtain

$$\frac{\Gamma_{(mag)}}{\Gamma} = \exp \left(\frac{3\pi B a \bar{q}^2}{(r_+ - M)} \right), \quad (70)$$

or

$$\frac{\Gamma_{(mag)}}{\Gamma} = \exp \left(\frac{3\pi B a \bar{q}^2}{\sqrt{M^2 - a^2 - \bar{q}^2}} \right). \quad (71)$$

where $\bar{q}^2 = q^2 + p^2$. The above equation shows that the tunneling probability of particles from the event horizon of magnetized Kerr-Newman black hole is greater than that in the absence of the magnetic field.

Magnetized Reissner-Nordström black hole

The tunneling probability of particles from magnetized Reissner-Nordström black hole can be obtained from the tunneling probability for magnetized Kerr-Newman black hole by setting $a = 0$, $p = 0$. In this case $r_+ = M + \sqrt{M^2 - q^2}$ and $\Omega_H = 0$. Thus from Eq. (35) we get

$$\Gamma_{(mag)} = \exp \left[-2\pi \frac{r_+^2}{r_+ - M} [E_s - e_s \Phi_0(r_+, \theta_0)] \right]_{|a=0, p=0}, \quad (72)$$

and from Eq. (31)

$$\Phi_0(r_+, \theta_0)|_{a=0} = \frac{\Phi_0^{(0)}(r_+, \theta_0) + B\Phi_0^{(1)}(r_+, \theta_0) + B^2\Phi_0^{(2)}(r_+, \theta_0) + B^3\Phi_0^{(3)}(r_+, \theta_0)}{4\Sigma(r_+, \theta_0)}, \quad (73)$$

with

$$\begin{aligned} \Phi_0^{(0)}(r_+, \theta_0) &= -4qr_+^3, \Phi_0^{(1)}(r_+, \theta_0) = 0, \\ \Phi_0^{(2)}(r_+, \theta_0) &= 3q[r_+^3(r_+^2 + 4Mr_+) + 6Mr_+^2 + 8M^2r_+], \Phi_0^{(3)}(r_+, \theta_0) = 0. \end{aligned}$$

Thus the tunneling probability from magnetized Reissner-Nordström black hole is

$$\Gamma_{(mag)} = \exp \left[-2\pi \frac{r_+^2}{r_+ - M} [E_s - e_s \{-4qr_+^3 + 3qB^2 (r_+^3(r_+^2 + 4Mr_+) + 6Mr_+^2 + 8M^2r_+)\}]. \right. \\ \left. \right. \quad (74)$$

In this case the Hawking temperature becomes

$$T_H = \frac{\sqrt{M^2 - q^2}}{2\pi \left(M + \sqrt{M^2 - q^2} \right)^2}. \quad (75)$$

The tunneling probability and temperature for Schwarzschild black hole can be recovered by setting $e_s = 0$ in Eq. (74) and Eq. (75) respectively.

Hawking temperature

The imaginary part of the action for the classically forbidden process is related to the Boltzmann factor for emission and the Hawking temperature. From (63) with $\Gamma = \exp[-\beta E]$, where $\beta = 1/T_H$ we find that the Hawking temperature is given by

$$T_H = \frac{(r_+ - M)}{2\pi(r_+^2 + a^2)}. \quad (5.1)$$

Using the value of r_+

$$T_H = \frac{\sqrt{M^2 - a^2 - q^2}}{2\pi \left(a^2 + (M + \sqrt{M^2 - a^2 - q^2})^2 \right)}, \quad (5.2)$$

We note that this temperature is the same as of the unmagnetized Kerr-Newman black hole which shows that background magnetic field does not effect the Hawking temperature of black hole.