

# Hawking radiation from magnetized black holes

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$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (1)$$

$$r_s = 2M \quad T_H = \kappa/2\pi \quad T_H = 1/8\pi M \quad (2)$$

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (3)$$

The electromagnetic four potential is given by

$$A^\alpha = \left(\frac{q}{r}, 0, 0, 0\right) \quad (4)$$

The event horizons are

$$r_{\pm} = M \pm \sqrt{M^2 - q^2} \quad (5)$$

## Kerr-Newman black hole

The Kerr-Newman geometry in Boyer and Lindquist coordinates has the form

$$ds^2 = -\frac{\Delta}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)d\phi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \quad (6)$$

where

$$\begin{aligned} \Delta &= r^2 - 2Mr + q^2 + a^2, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta, \\ a &= \frac{J}{M}, \end{aligned}$$

where  $J$  is the angular momentum of the black hole. The electromagnetic vector potential can be written as

$$A = (\Phi_0 - \omega \Phi_3) dt + \Phi_3 d\phi,$$

with

$$\Phi_0 = -\frac{qr(r^2 + a^2)}{\rho^2}, \quad \Phi_3 = \frac{aqr \sin^2 \theta}{\rho^2}.$$

This black hole has two horizons given by

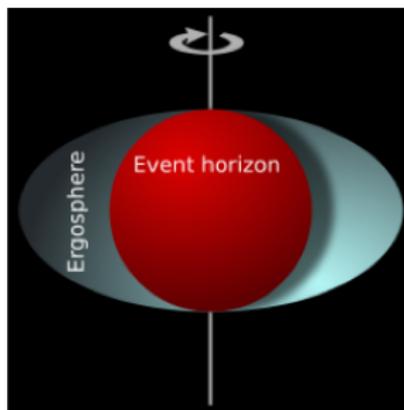
$$r^2 - 2Mr + q^2 + a^2 = 0.$$

Its solutions are

$$r_{\pm} = M \pm \sqrt{M^2 - q^2 - a^2}, \quad (7)$$

which are the expressions for the location of inner and outer horizons. Hawking temperature is

$$T_H = \frac{\sqrt{M^2 - a^2 - q^2}}{2\pi \left( a^2 + (M + \sqrt{M^2 - a^2 - q^2})^2 \right)}. \quad (8)$$



The one-parameter static solution of the coupled Einstein–Maxwell system is given by the metric

$$ds^2 = \left(1 + \frac{1}{4}B^2\rho^2\right)^2(-dt^2 + d\rho^2 + dz^2) + \left(1 + \frac{1}{4}B^2\rho^2\right)^{-2}\rho^2 d\phi^2 \quad (9)$$

with  $t, z \in (-\infty, +\infty)$ ,  $\rho \in [0, \infty)$ ,  $\phi \in [0, 2\pi)$ . The electromagnetic field can be described by the Maxwell tensor

$$F = e^{-i\psi} B(dz \wedge dt) \quad (10)$$

where  $\psi$  is real parameter of duality rotation. In particular, for  $\psi = 0$ , the Maxwell tensor is  $F = Bdz \wedge dt$  which describes an electric field pointing along the  $z$ -direction, whereas for  $\psi = \pi/2$  one obtains  $F = B(1 + 1/4B^2\rho^2)^{-2}\rho d\rho \wedge d\phi$ , which represents a purely magnetic field oriented along the  $z$ -direction.

## Harrison's transformation and Ernst's technique

In 1976, F. J. Ernst using Harrison transformation presented a procedure which was used for transforming asymptotically flat axially symmetric solutions of the coupled Einstein-Maxwell equations into solution resembling Melvin's magnetic universe. He used this technique for the removal of the nodal singularity of the  $C$ -metric and studied the Schwarzschild, Reissner-Nordström and Kerr black holes in Melvin universe.

## Schwarzschild black hole in Melvin universe

The Harrison transformations can be used to magnetize the Schwarzschild black hole. The line element becomes

$$ds^2 = \left(1 + \frac{1}{4}B^2 r^2 \sin^2 \theta\right)^2 \left[-\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2\right] \quad (11)$$

$$+ \left(1 + \frac{1}{4}B^2 r^2 \sin^2 \theta\right)^{-2} r^2 \sin^2 \theta d\phi^2 \quad (12)$$

In this case the magnetic field components are given by

$$H_r = \Lambda^{-2} B \cos \theta \quad (13)$$

$$H_\theta = -\Lambda^{-2} B \left(1 - \frac{2M}{r}\right)^{1/2} \sin \theta \quad (14)$$

$$\Lambda = 1 + B\Phi - \frac{1}{4}B^2 \varepsilon = 1 + \frac{1}{4}B^2 r^2 \sin^2 \theta, \quad (15)$$

$$(16)$$

Note that if  $M = 0$  the above metric becomes the Melvin's magnetic universe, while for  $M \neq 0$  there is an event horizon at  $r = 2M$  and the angular component of magnetic field vanishes at the event horizon. Further, the metric has singularity at  $r = 0$ , as in case of Schwarzschild metric.

## Reissner-Nordström black hole in Melvin universe

The Reissner-Nordström black hole in Melvin universe is

$$ds^2 = (\Lambda)^2 \left[ - \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{q^2}{r^2} \right)^{-1} dr^2 + r^2 d\theta^2 \right] \quad (17)$$

$$+ (\Lambda)^{-2} r^2 \sin^2 \theta (d\phi - \omega' dt)^2 \quad (18)$$

where

$$\Lambda = 1 + B\Phi - \frac{1}{4}B^2 = 1 + \frac{1}{4}B^2(r^2 \sin^2 \theta + q^2 \cos^2 \theta) - iBq \cos \theta, \quad (19)$$

$$(20)$$

If  $q = 0$  then this metric reduces to the Schwarzschild black hole in Melvin universe, and if  $B = 0$ , then this becomes Reissner-Nordström.

The components of the electric and magnetic fields from the electromagnetic potential  $\Phi$  are

$$H_r + iE_r = \Lambda^{-2} [i(\frac{e}{r^2}) \{1 - \frac{1}{4} B^2 (r^2 \sin^2 \theta + q^2 \cos^2 \theta)\}] \quad (21)$$

$$+ B(1 - \frac{1}{2} iBq \cos \theta)(1 - \frac{q^2}{r^2}) \cos \theta], \quad (22)$$

and

$$H_\theta + iE_\theta = -B |\Lambda|^2 (1 - \frac{1}{2} iq^2 \cos \theta)(1 - \frac{2M}{r} + \frac{q^2}{r^2})^{1/2} \sin \theta.$$

## Magnetized Kerr-Newman black hole

The magnetized Kerr-Newman black hole of mass  $M$ , angular momentum per unit mass  $a$ , carrying an electric charge  $q$ , embedded in a uniform background magnetic field  $B$  is

$$ds^2 = H[-fdt^2 + R^2(\frac{dr^2}{\Delta} + d\theta^2)] + \frac{\Sigma \sin^2 \theta}{HR^2}(d\phi - \omega dt)^2, \quad (23)$$

where

$$R^2 = r^2 + a^2 \cos^2 \theta, \quad (24)$$

$$\Delta = (r^2 + a^2) - 2Mr + q^2, \quad (25)$$

$$\Sigma = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad (26)$$

$$f = \frac{R^2 \Delta}{\Sigma}, \quad (27)$$

$$H = 1 + \frac{H_{(1)}B + H_{(2)}B^2 + H_{(3)}B^3 + H_{(4)}B^4}{R^2}, \quad (28)$$

with

$$H_{(1)} = 2aqr \sin^2 \theta - 2p(r^2 + a^2) \cos \theta,$$

$$H_{(2)} = \frac{1}{2}[(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] \sin^2 \theta + \frac{3}{2}q^2(a^2 + r^2 \cos^2 \theta),$$

$$H_{(3)} = -\frac{qa\Delta}{2r}[r^2(3 - \cos^2 \theta) \cos^2 \theta + a^2(1 + \cos^2 \theta)] + \frac{1}{2}p(r^4 - a^4) \sin^2 \theta \cos \theta \\ + \frac{q\bar{q}a[(2r^2 + a^2) \cos^2 \theta + a^2]}{2r} - pa^2 \Delta \sin^2 \theta \cos \theta - \frac{1}{2}p\bar{q}^2(r^2 + a^2) \cos^3 \theta \\ + \frac{aq(r^2 + a^2)^2(1 + \cos^2 \theta)}{2r},$$

$$H_{(4)} = \frac{1}{16}(r^2 + a^2)^2 R^2 \sin^4 \theta + \frac{1}{4}M^2 a^2 [r^2(\cos^2 \theta - 3)^2 \cos^2 \theta + a^2(1 + \cos^2 \theta)^2] \\ + \frac{1}{4}Ma^2 r(r^2 + a^2) \sin^6 \theta + \frac{1}{16}\bar{q}^4 [r^2 \cos^2 \theta + a^2(1 + \sin^2 \theta)] \cos^2 \theta \\ + \frac{1}{4}Ma^2 r(r^2 + a^2) \sin^6 \theta + \frac{1}{4}Ma^2 \bar{q}^2 r(\cos^2 \theta - 5) \sin^2 \theta \cos^2 \theta \\ + \frac{1}{8}\bar{q}^2 (r^2 + a^2)(r^2 + a^2 + a^2 \sin^2 \theta) \sin^2 \theta \cos^2 \theta.$$

Here

$$\bar{q}^2 = q^2 + p^2,$$

and

$$\omega = \frac{1}{\Sigma} [(2Mr - \bar{q}^2)a + \omega_{(1)}B + \omega_{(2)}B^2 + \omega_{(3)}B^3 + \omega_{(4)}B^4], \quad (29)$$

with

$$\omega_{(1)} = -2qr(r^2 + a^2) + 2ap\Delta \cos \theta,$$

$$\omega_{(2)} = -\frac{3}{2}a\bar{q}^2(r^2 + a^2 + \Delta \cos^2 \theta),$$

$$\begin{aligned} \omega_{(3)} = & 4qM^2a^2r + \frac{1}{2}ap\bar{q}^4 \cos^3 \theta + \frac{1}{2}qr(r^2 + a^2)[r^2 - a^2 + (r^2 + 3a^2) \cos^2 \theta] \\ & + \frac{1}{2}ap(r^2 + a^2)[3r^2 + a^2 - (r^2 - a^2) \cos^2 \theta] \cos \theta - aM\bar{q}^2(2aq + pr \cos^3 \theta) \\ & - apMr[2R^2 + (r^2 + a^2) \sin^2 \theta] \cos \theta + \frac{1}{2}ap\bar{q}^2[3r^2 + a^2 + 2a^2 \cos^2 \theta] \cos \theta \\ & + \frac{1}{2}q\bar{q}^2r[(r^2 + 3a^2) \cos^2 \theta - 2a^2] + qM[r^4 - a^4 + r^2(r^2 + 3a^2) \sin^2 \theta], \end{aligned}$$

$$\begin{aligned} \omega_{(4)} = & \frac{1}{2}a^3M^3r(3 + \cos^4 \theta) - \frac{1}{8}a\bar{q}^4[r^2(2 + \sin^2 \theta) \cos^2 \theta + a^2(1 + \cos^4 \theta)] \\ & + \frac{1}{16}a\bar{q}^2(r^2 + a^2)[r^2(1 - 6 \cos^2 \theta + 3 \cos^4 \theta) - a^2(a + \cos^4 \theta)] \\ & - \frac{1}{4}a^3M^2\bar{q}^2(3 + \cos^4 \theta) - \frac{1}{16}a\bar{q}^6 \cos^4 \theta + \frac{1}{4}aM^2[r^4(3 - 6 \cos^2 \theta + 3 \cos^4 \theta) \\ & + 2a^2r^2(3 \sin^2 \theta - 2 \cos^4 \theta) - a^4(1 + \cos^4 \theta)] + \frac{1}{2}aM\bar{q}^4r \cos^4 \theta \end{aligned}$$

The electromagnetic vector potential is

$$A = (\Phi_0 - \omega\Phi_3)dt + \Phi_3 d\phi, \quad (30)$$

where

$$\Phi_0 = \frac{\Phi_0^{(0)} + \Phi_0^{(1)}B + \Phi_0^{(2)}B^2 + \Phi_0^{(3)}B^3}{4\Sigma}, \quad (31)$$

with

$$\begin{aligned} \Phi_0^{(0)} &= 4[-qr(r^2 + a^2) + ap\Delta \cos \theta], \\ \Phi_0^{(1)} &= -6a\bar{q}^2(r^2 + a^2 + \Delta \cos^2 \theta), \\ \Phi_0^{(2)} &= -3q[(r + 2M)a^4 - (r^2 + 4Mr + \Delta \cos^2 \theta)r^3 + a^2(2\bar{q}^2(r + 2M) - 6Mr^2 \\ &\quad - 8M^2r - 3r\Delta \cos^2 \theta)] + 3p\Delta[3ar^2 + a^3 + a(a^2 + \bar{q}^2 - r^2) \cos^2 \theta] \cos \theta, \\ \Phi_0^{(3)} &= -\frac{1}{2}a[4a^4M^2 + 12a^2M^2\bar{q}^2 + 2a^2\bar{q}^4 + 2a^4Mr - 24a^2M^3r + 4a^2M\bar{q}^2r \\ &\quad - 24a^2M^2r^2 - 4a^2Mr^3 - \bar{q}^2r^4 - 6Mr^5 - 6r\Delta\{2M(r^2 + a^2) - \bar{q}^2r\} \cos^2 \theta \\ &\quad + a^4\bar{q}^2 - 12M^2r^4 + \Delta(\bar{q}^4 - 3\bar{q}^2r^2 + a^2(4M^2 + \bar{q}^2 - 6Mr)) \cos^4 \theta], \end{aligned}$$

and

$$\Phi_3 = \frac{\Phi_3^{(0)} + \Phi_3^{(1)} + \Phi_3^{(2)} + \Phi_3^{(3)}}{R^2 H}, \quad (32)$$

with

$$\Phi_3^{(0)} = aqr \sin^2 \theta - p(r^2 + a^2) \cos \theta,$$

$$\Phi_3^{(1)} = \frac{1}{2} [\Sigma \sin^2 \theta + 3\bar{q}^2 (a^2 + r^2 \cos^2 \theta)],$$

$$\begin{aligned} \Phi_3^{(2)} = & \frac{3}{4} aqr (r^2 + a^2) \sin^4 \theta - \frac{3}{4} p (r^2 + a^2)^2 \sin^2 \theta \cos \theta + 3a^2 pMr \sin^2 \theta \cos \theta \\ & + \frac{3}{2} aqm [r^2 (3 - \cos^2 \theta) \cos^2 \theta + a^2 (1 + \cos^2 \theta)] - \frac{3}{4} aq\bar{q}^2 r \sin^2 \theta \cos^2 \theta \\ & - \frac{3}{4} p\bar{q}^2 [(r^2 - a^2) \cos^2 \theta + 2a^2] \cos \theta, \end{aligned}$$

$$\begin{aligned} \Phi_3^{(3)} = & \frac{1}{4} \bar{q}^2 (r^2 + a^2) [r^2 + a^2 + a^2 \sin^2 \theta \cos^2 \theta] - \frac{1}{2} a^2 \bar{q}^2 Mr (5 - \cos^2 \theta) \sin^2 \theta \cos^2 \theta \\ & + \frac{1}{2} a^2 M^2 [r^2 (3 - \cos^2 \theta)^2 \cos^2 \theta + a^2 (1 + \cos^2 \theta)^2] + \frac{1}{2} a^2 Mr (r^2 + a^2) \sin^6 \theta \\ & + \frac{1}{8} R^2 (r^2 + a^2)^2 \sin^4 \theta + \frac{1}{8} \bar{q}^4 [r^2 \cos^2 \theta + a^2 (2 - \cos^2 \theta)^2] \cos^2 \theta. \end{aligned}$$

## Quantum tunneling of charged scalar particles

We discuss quantum tunneling of charged scalar particles from event horizon of the magnetized Kerr-Newman black hole using Hamilton Jacobi method which is a semi-classical approach.

First we write the metric in the form

$$ds^2 = -\left(fH - \frac{\Sigma \sin^2 \theta \omega^2}{HR^2}\right) dt^2 + \frac{HR^2}{\Delta} dr^2 + HR^2 d\theta^2 + \frac{\Sigma \sin^2 \theta}{HR^2} d\phi^2 - 2 \frac{\Sigma \sin^2 \theta \omega}{HR^2} dt d\phi \quad (33)$$

Using new functions  $C(r, \theta)$ ,  $g(r, \theta)$ ,  $D(r, \theta)$ ,  $K(r, \theta)$  and  $G(r, \theta)$  this becomes

$$ds^2 = -C(r, \theta) dt^2 + \frac{dr^2}{g(r, \theta)} + D(r, \theta) d\theta^2 + K(r, \theta) d\phi^2 - 2G(r, \theta) dt d\phi, \quad (34)$$

$$C(r, \theta) = fH - \frac{\Sigma \sin^2 \theta \omega^2}{HR^2},$$

$$g(r, \theta) = \frac{\Delta}{HR^2},$$

$$D(r, \theta) = HR^2,$$

$$K(r, \theta) = \frac{\Sigma \sin^2 \theta}{HR^2},$$

$$G(r, \theta) = \frac{\Sigma \sin^2 \theta \omega}{HR^2}.$$

The event horizons of the magnetized Kerr-Newman black hole is obtained from

$$(g_{rr})^{-1} = g(r, \theta) = \frac{\Delta}{HR^2}.$$

The outer and inner horizons corresponding to this black hole denoted by " $r_{\pm}$ " are given by

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - q^2 - p^2}. \quad (35)$$

The angular velocity at black hole horizon is

$$\Omega_H = \frac{G(r_+, \theta)}{K(r_+, \theta)}. \quad (36)$$

Using the values of  $G(r_+, \theta)$  and  $K(r_+, \theta)$  this becomes

$$\Omega_H = \omega(r_+, \theta). \quad (37)$$

We define new function

$$F(r, \theta) = -(g^{tt})^{-1} = C(r, \theta) + \frac{G^2(r, \theta)}{K(r, \theta)}. \quad (38)$$

Using the value of  $C(r, \theta)$ ,  $G(r, \theta)$  and  $K(r, \theta)$  we will get

$$F(r, \theta) = Hf. \quad (39)$$

Now we shall solve the Klein-Gordon equation to study the quantum tunneling of charged scalar particles from event horizons of magnetized Kerr-Newman black hole. The Klein-Gordon equation for scalar field  $\Phi$  is

$$g^{\alpha\beta} \left( \partial_\alpha - \frac{ie_s}{\hbar} A_\alpha \right) \left( \partial_\beta - \frac{ie_s}{\hbar} A_\beta \right) \Phi - \frac{m_s^2}{\hbar^2} \Phi = 0, \quad (40)$$

where  $\alpha, \beta = 1, 2, 3, 4$  corresponds to  $t, r, \theta, \phi$  respectively,  $e_s$  is the charge,  $m_s$  is its mass,  $g^{\alpha\beta}$  is the inverse of the metric tensor and  $A_\alpha$  is the vector potential which is given by Eq. (30).

Applying the Wentzel-Kramen-Brillouin (WKB) approximation and assuming the following ansatz

$$\Phi(t, r, \theta, \phi) = \exp\left[\frac{i}{\hbar}I(t, r, \theta, \phi) + I_1(t, r, \theta, \phi) + O(\hbar)\right], \quad (41)$$

where  $I$  is the action.

After simplifying we get

$$\begin{aligned}
 \left(\partial_t - \frac{ie_s}{\hbar} A_t\right)^2 \Phi &= \exp\left(\frac{i}{\hbar} I + I_1\right) \left[ \left(\frac{i}{\hbar} \partial_t I + \partial_t I_1\right)^2 + \left(\frac{i}{\hbar} \partial_{tt} I + \partial_{tt} I_1\right) - \frac{ie_s}{\hbar} \partial_t A_t \right. \\
 &\quad \left. - \frac{ie_s}{\hbar} A_t \left(\frac{i}{\hbar} \partial_t I + \partial_t I_1\right) - \frac{ie_s}{\hbar} A_t \left(\frac{i}{\hbar} \partial_t I + \partial_t I_1\right) + \left(\frac{ie_s A_t}{\hbar}\right)^2 \right]. \quad (42)
 \end{aligned}$$

$$\begin{aligned}
 \left(\partial_r - \frac{ie_s}{\hbar} A_r\right)^2 \Phi &= \exp\left(\frac{i}{\hbar} I + I_1\right) \left[ \left(\frac{i}{\hbar} \partial_r I + \partial_r I_1\right)^2 + \left(\frac{i}{\hbar} \partial_{rr} I + \partial_{rr} I_1\right) - \frac{ie_s}{\hbar} \partial_r A_r \right. \\
 &\quad \left. - \frac{ie_s}{\hbar} A_r \left(\frac{i}{\hbar} \partial_r I + \partial_r I_1\right) - \frac{ie_s}{\hbar} A_r \left(\frac{i}{\hbar} \partial_r I + \partial_r I_1\right) + \left(\frac{ie_s A_r}{\hbar}\right)^2 \right], \quad (43)
 \end{aligned}$$

$$\begin{aligned}
 \left(\partial_\theta - \frac{ie_s}{\hbar} A_\theta\right)^2 \Phi &= \exp\left(\frac{i}{\hbar} I + I_1\right) \left[ \left(\frac{i}{\hbar} \partial_\theta I + \partial_\theta I_1\right)^2 + \left(\frac{i}{\hbar} \partial_{\theta\theta} I + \partial_{\theta\theta} I_1\right) - \frac{ie_s}{\hbar} \partial_\theta A_\theta \right. \\
 &\quad \left. - \frac{ie_s}{\hbar} A_\theta \left(\frac{i}{\hbar} \partial_\theta I + \partial_\theta I_1\right) - \frac{ie_s}{\hbar} A_\theta \left(\frac{i}{\hbar} \partial_\theta I + \partial_\theta I_1\right) + \left(\frac{ie_s A_\theta}{\hbar}\right)^2 \right], \quad (44)
 \end{aligned}$$

$$\begin{aligned}
(\partial_\phi - \frac{ie_s}{\hbar} A_\phi)^2 \Phi &= \exp(\frac{i}{\hbar} I + I_1) [(\frac{i}{\hbar} \partial_t I + \partial_t I_1)^2 + (\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) - \frac{ie_s}{\hbar} \partial_t A_\phi \\
&\quad - \frac{ie_s}{\hbar} A_\phi (\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) - \frac{ie_s}{\hbar} A_t (\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) + (\frac{ie_s A_\phi}{\hbar})^2], \quad (45)
\end{aligned}$$

and

$$\begin{aligned}
(\partial_t - \frac{ie_s}{\hbar} A_t)(\partial_\phi - \frac{ie_s}{\hbar} A_\phi) \Phi &= \exp(\frac{i}{\hbar} I + I_1) [(\frac{i}{\hbar} \partial_t I + \partial_t I_1)(\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) \\
&\quad + (\frac{i}{\hbar} \partial_{t\phi} I + \partial_{t\phi} I_1) - \frac{ie_s}{\hbar} A_\phi (\frac{i}{\hbar} \partial_t I + \partial_t I_1) \\
&\quad - \frac{ie_s}{\hbar} \partial_t A_\phi - \frac{ie_s}{\hbar} A_t (\frac{i}{\hbar} \partial_\phi I + \partial_\phi I_1) + (\frac{ie_s}{\hbar})^2 A_t A_\phi]. \quad (46)
\end{aligned}$$

The above equation can also written as

$$g^{\alpha\beta}(\partial_\alpha I - e_s A_\alpha)(\partial_\beta I - e_s A_\beta) + m_s^2 = 0, \quad (47)$$

which after using the values of  $g^{\alpha\beta}$  where  $\alpha, \beta = 1, 2, 3, 4$ , and simplification reduces to the following equation

$$g(r, \theta)(\partial_r I)^2 - \frac{2H(r, \theta)}{F(r, \theta)K(r, \theta)}(\partial_t I - e_s A_t)(\partial_\phi I - e_s A_\phi) - \frac{(\partial_t I - e_s A_t)^2}{F(r, \theta)} \quad (48)$$

$$- \frac{(\partial_t I - e_s A_t)^2}{F(r, \theta)} + \frac{C(r, \theta)}{F(r, \theta)K(r, \theta)}(\partial_\phi I - e_s A_\phi)^2 + \frac{(\partial_\theta I)^2}{(D(r, \theta))^2} + m_s^2 = 0.$$

For the calculation of tunneling probability consider an ansatz of the form

$$I = -tE_s + \phi J_s + W(r, \theta), \quad (49)$$

where  $E_s$  and  $J_s$  are energy and angular momentum of scalar particles.

For fixed value of  $\theta = \theta_0 = 0$ , we have

$$I = -tE_s + \phi J_s + S(r) + \Theta(\theta_0). \quad (50)$$

Using these in Eq. (48) we have

$$\begin{aligned} & \frac{2H(r, \theta_0)}{F(r, \theta_0)K(r, \theta_0)}(E_s + e_s A_t)(J_s - e_s A_\phi) + \frac{C(r, \theta_0)}{F(r, \theta_0)K(r, \theta_0)}(J_s - e_s A_\phi)^2 \\ & + g(r, \theta_0)(S'(r))^2 - \frac{(E_s + e_s A_t)^2}{F(r, \theta_0)} + m_s^2 = 0. \end{aligned} \quad (51)$$

Using Taylor's theorem we obtain

$$F(r, \theta_0) = (r - r_+)F_r(r_+, \theta_0) = \frac{2(r - r_+)(r_+ - M)H(r_+)R^2(r_+)}{\Sigma(r_+, \theta_0)}, \quad (52)$$

$$g(r, \theta_0) = (r - r_+)g_r(r_+, \theta_0) = \frac{2(r - r_+)(r_+ - M)}{H(r_+)(r_+^2 + a^2 \cos^2 \theta_0)}, \quad (53)$$

$$\Omega_H = \frac{G(r_+, \theta_0)}{K(r_+, \theta_0)} = \omega(r_+, \theta_0). \quad (54)$$

Since the above equation is quadratic in terms of  $S(r)$ , so we have two solutions, one solution corresponds to scalar particles moving away from the black hole and the other solution corresponds to particles moving towards the black hole. Thus

$$S_{\pm}(r) = \pm \int \frac{(r_+^2 + a^2)dr}{2(r_+ - M)(r - r_+)} \times \sqrt{(E_s - \Omega_H J_s - e_s \Phi_o)^2 - \frac{2H(r_+)(r_+ - M)(r - r_+)(r_+^2 + a^2 \cos^2 \theta_0)}{(r_+^2 + a^2)^2} \left[ \frac{(J_s - e_s A_\phi)^2}{K(r_+, \theta_0)} + m_s^2 \right]}. \quad (55)$$

Integrating the above equation using residue theory we will get

$$ImS_+ = \frac{\pi}{2} \frac{(E_s - \Omega_H J_s - e_s \Phi_o(r_+, \theta_0))(r_+^2 + a^2)}{(r_+ - M)}, \quad (56)$$

and

$$ImS_- = -\frac{\pi}{2} \frac{(E_s - \Omega_H J_s - e_s \Phi_o(r_+, \theta_0))(r_+^2 + a^2)}{(r_+ - M)}. \quad (57)$$

We note that

$$ImS_+ = -ImS_-. \quad (58)$$

The tunneling probabilities of particles in each direction are

$$P_{out} = \exp[-2ImI] = \exp[-2(ImS_+ + Im\Theta)], \quad (59)$$

$$P_{in} = \exp[-2ImI] = \exp[-2(ImS_- + Im\Theta)]. \quad (60)$$

Thus the resulting tunneling probability  $\Gamma_{(mag)}$  of a particle from inside to outside the horizon of magnetized Kerr-Newman black hole is

$$\Gamma_{(mag)} \propto \frac{P_{out}}{P_{in}} = \frac{\exp[-2(ImS_+ + Im\Theta)]}{\exp[-2(ImS_- + Im\Theta)]},$$

or

$$\Gamma_{(mag)} = \exp[-2(ImS_+ - ImS_-)]. \quad (61)$$

$$\Gamma_{(mag)} = \exp[-4ImS_+], \quad (62)$$

which after using Eq. (56) the tunneling probability of scalar particle becomes

$$\Gamma_{(mag)} = \exp \left[ \frac{-2\pi (r_+^2 + a^2)}{(r_+ - M)} (E_s - \Omega_H J_s - e_s \Phi_o(r_+, \theta_0)) \right]. \quad (63)$$

Note that the tunneling probability depends upon the mass  $M$ , angular momentum per unit mass  $a$ , the electric charge  $q$  of the black hole, as well as charge  $e_s$ , the energy  $E_s$ , and the angular momentum  $J_s$  of the scalar particles and the background magnetic field  $B$ .

## In the absence of magnetic field i.e. for $B = 0$

The tunneling probability of scalar particle from the event horizon of magnetized Kerr-Newman black hole given by Eq. (35), in the absence of magnetic field i.e. for  $B = 0$ , should give us the tunneling probability of Kerr-Newman black hole. Under this condition Eqs. (31) imply

$$\Phi_o(r_+, \theta_0) = \frac{\Phi_o^{(0)}(r_+, \theta_0)}{4\Sigma(r_+, \theta_0)}, \quad (64)$$

or

$$\Phi_o(r_+, \theta_0) = \frac{-4qr_+(r_+^2 + a^2)}{4(r_+^2 + a^2)^2} = \frac{-qr_+}{(r_+^2 + a^2)}, \quad (65)$$

so that Eqs. (35) yields

$$\Gamma = \exp \left[ \frac{-2\pi (r_+^2 + a^2)}{(r_+ - M)} \left( E_s - \Omega_H J_s - \frac{e_s qr_+}{(r_+^2 + a^2)} \right) \right], \quad (66)$$

which is the tunneling probability of Kerr-Newman black hole as given in the literature [?].

If we consider the situation where the magnetic field is weak i.e.  $B^n = 0$  for  $n \geq 2$ , Eq. (31) becomes

$$\Phi_0(r_+, \theta_0) = \frac{\Phi_0^{(0)}(r_+, \theta_0) + B\Phi_0^{(1)}(r_+, \theta_0)}{4\Sigma(r_+, \theta_0)}. \quad (67)$$

Or

$$\Phi_0(r_+, \theta_0) = -\frac{(4qr_+ + 6Ba\bar{q}^2)}{4(r_+^2 + a^2)}. \quad (68)$$

So the tunneling probability in the weak magnetic field is obtained as

$$\Gamma_{(mag)} = \exp \left[ \frac{-2\pi (r_+^2 + a^2)}{(r_+ - M)} \left( E_s - \Omega_H J_s - \frac{e_s q r_+}{(r_+^2 + a^2)} \right) \right] \exp \left( \frac{3\pi B a \bar{q}^2}{(r_+ - M)} \right). \quad (69)$$

Using Eq. (66) we obtain

$$\frac{\Gamma_{(mag)}}{\Gamma} = \exp\left(\frac{3\pi Ba\bar{q}^2}{(r_+ - M)}\right), \quad (70)$$

or

$$\frac{\Gamma_{(mag)}}{\Gamma} = \exp\left(\frac{3\pi Ba\bar{q}^2}{\sqrt{M^2 - a^2 - \bar{q}^2}}\right). \quad (71)$$

where  $\bar{q}^2 = q^2 + p^2$ . The above equation shows that the tunneling probability of particles from the event horizon of magnetized Kerr-Newman black hole is greater than that in the absence of the magnetic field.

# Magnetized Reissner-Nordström black hole

The tunneling probability of particles from magnetized Reissner-Nordström black hole can be obtained from the tunneling probability for magnetized Kerr-Newman black hole by setting  $a = 0$ ,  $p = 0$ . In this case  $r_+ = M + \sqrt{M^2 - q^2}$  and  $\Omega_H = 0$ . Thus from Eq. (35) we get

$$\Gamma_{(mag)} = \exp \left[ -2\pi \frac{r_+^2}{r_+ - M} [E_s - e_s \Phi_0(r_+, \theta_0)] \right]_{|a=0, p=0}, \quad (72)$$

and from Eq. (31)

$$\Phi_0(r_+, \theta_0)|_{a=0} = \frac{\Phi_0^{(0)}(r_+, \theta_0) + B\Phi_0^{(1)}(r_+, \theta_0) + B^2\Phi_0^{(2)}(r_+, \theta_0) + B^3\Phi_0^{(3)}(r_+, \theta_0)}{4\Sigma(r_+, \theta_0)}, \quad (73)$$

with

$$\begin{aligned} \Phi_0^{(0)}(r_+, \theta_0) &= -4qr_+^3, \quad \Phi_0^{(1)}(r_+, \theta_0) = 0, \\ \Phi_0^{(2)}(r_+, \theta_0) &= 3q[r_+^3(r_+^2 + 4Mr_+) + 6Mr_+^2 + 8M^2r_+], \quad \Phi_0^{(3)}(r_+, \theta_0) = 0. \end{aligned}$$

Thus the tunneling probability from magnetized Reissner-Nordström black hole is

$$\Gamma_{(mag)} = \exp \left[ -2\pi \frac{r_+^2}{r_+ - M} [E_s - e_s \{-4qr_+^3 + 3qB^2 (r_+^3(r_+^2 + 4Mr_+) + 6Mr_+^2 + 8M^2r_+)\}] \right]. \quad (74)$$

In this case the Hawking temperature becomes

$$T_H = \frac{\sqrt{M^2 - q^2}}{2\pi \left( M + \sqrt{M^2 - q^2} \right)^2}. \quad (75)$$

The tunneling probability and temperature for Schwarzschild black hole can be recovered by setting  $e_s = 0$  in Eq. (74) and Eq. (75) respectively.

## Hawking temperature

The imaginary part of the action for the classically forbidden process is related to the Boltzmann factor for emission and the Hawking temperature. From (63) with  $\Gamma = \exp[-\beta E]$ , where  $\beta = 1/T_H$  we find that the Hawking temperature is given by

$$T_H = \frac{(r_+ - M)}{2\pi (r_+^2 + a^2)}. \quad (5.1)$$

Using the value of  $r_+$

$$T_H = \frac{\sqrt{M^2 - a^2 - q^2}}{2\pi \left( a^2 + (M + \sqrt{M^2 - a^2 - q^2})^2 \right)}, \quad (5.2)$$

We note that this temperature is the same as of the unmagnetized Kerr-Newman black hole which shows that background magnetic field does not effect the Hawking temperature of black hole.