Strong Cosmic Censorship in cosmological Bianchi class B perfect fluids and vacuum

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Overview

- Einstein's field equations and the Strong Cosmic Censorship conjecture
- Our setting: Bianchi B models and perfect fluids
- Known and new results
- Techniques and method of proof

Einstein's equation as initial value problem

Given: Riemannian manifold (S, h) and two-tensor k on S such that

$$R(h) - |k|_h^2 + (\mathrm{tr}_h\, k)^2 = 2\mu_0, \qquad \text{(Hamilton constr.)}$$

$$\mathrm{div}_h\, k - \nabla(\mathrm{tr}_h\, k) = 0. \qquad \text{(Momentum constr.)}$$

Find spacetime (M, g) solving

$$Ein = Ric - \frac{1}{2}Rg = 8\pi T \qquad \text{(Einstein's equ.)}$$

for chosen type of matter and correct initial conditions.

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Theorem (Choquet-Bruhat and Geroch 1969 (vacuum case))

There exists a maximal globally hyberbolic spacetime (M, g) with these conditions where $S \to M$ is a Cauchy hypersurface. Up to isometry, the spacetime (M, g) is unique, it is called the maximal globally hyperbolic development.

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Conjecture (Strong Cosmic Censorship)

Given generic initial data, the maximal globally hyperbolic development is inextendible.

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- restrict to specific matter models, here Bianchi B perfect fluids

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Given generic initial data, the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded in the incomplete directions of causal geodesics in the maximal globally hyperbolic development.

Curvature blow-up conjecture $\Rightarrow$ Strong Cosmic Censorship in C^2 sense.

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Curvature blow-up conjecture $\Rightarrow$ Strong Cosmic Censorship in C^2 sense.
In case $Ric \neq 0$, consider scalar $Ric_{\alpha\beta}Ric^{\alpha\beta}$ instead of Kretschmann scalar.

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      - early work by Chruściel–Rendall '95
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      - e. g. Chruściel–Isenberg–Moncrief '90, Ringström '09
  - Asymptotically flat
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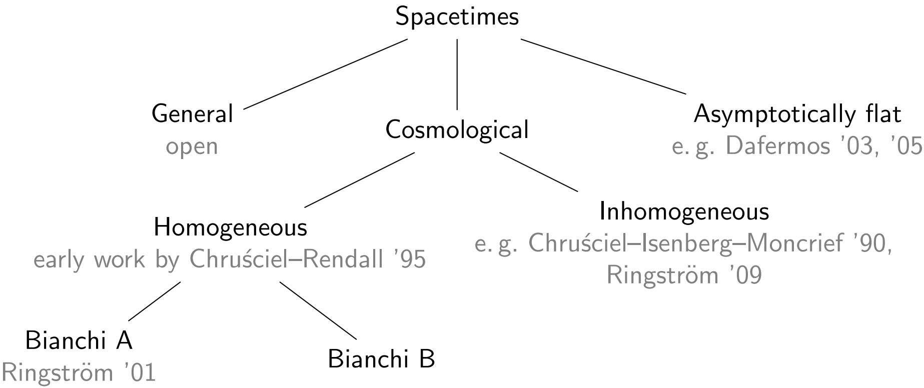

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      - Bianchi A
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      - Bianchi B
        - Orthogonal
          - topic of this talk
        - Exceptional
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Our setting I: Cosmological Bianchi models

Cosmological spacetimes: Cauchy surface S is compact without boundary.
For our setting: $S = G$ *three-dimensional Lie group*, and find metric on $M = \mathbb{R} \times G$ which is left-invariant on every $\{t\} \times G$.

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Classification of three-dimensional Lie groups following [Bianchi 1898]:

- class A (G is unimodular): types I, II, VI_0, VII_0, VIII, IX
- class B (G is non-unimodular): types IV, V, VI_η, VII_η. The topic of this talk, exclude the exceptional Bianchi type $\mathrm{VI}_{-1/9}$.

Our setting II: Perfect fluids

Perfect fluid: spacetime (M, g) satisfying Einstein's equation with stress-energy tensor

$$8\pi T_{\alpha\beta} = \mu u_\alpha u_\beta + p(g + u_\alpha u_\beta)$$

for a unit timelike vector field u. Assume: pressure p and energy density μ satisfy linear equation of state

$$p = (\gamma - 1)\mu$$

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We construct *non-tilted* spacetimes, i. e. with u normal to the Cauchy hypersurfaces $\{t\} \times G$.

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Theorem (R. tbp)

Consider orthogonal perfect fluid Bianchi class B initial data (G, h, k, μ_0) *for matter* $\mu_0 > 0$*. Let* (M, g) *be the maximal globally hyperbolic development, solving Einstein's equations for a perfect fluid with linear equation of state, where* $\gamma > 0$*. Then the contraction of the Ricci tensor with itself* $R_{\alpha\beta}R^{\alpha\beta}$ *is unbounded in the incomplete directions of causal geodesics.*

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Theorem (R. tbp)

Consider orthogonal perfect fluid Bianchi class B initial data (G, h, k, μ_0) for vacuum $\mu_0 = 0$, which is neither locally rotationally symmetric of Bianchi type III nor of plane wave equilibrium type. Let (M, g) be the maximal globally hyperbolic development, solving Einstein's vacuum equations. Then the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded in the incomplete directions of causal geodesics.

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Excluded spacetimes have additional symmetry $\Rightarrow$ can be considered non-generic. As a consequence, **Strong Cosmic Censorship holds in the class of orthogonal perfect fluid and vacuum Bianchi class B initial data.**

Techniques

- Transform initial data into new set of variables, contained in a compact set of $\mathbb{R}^5$.
  Einstein's field equations transform into polynomial evolution equations for these variables with two polynomial constraint equations [Hewitt-Wainwright 1993].
- Discuss all Bianchi B types simultaneously (but exclude Bianchi type $\mathrm{VI}_{-1/9}$).

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Thank you for your attention!

Special initial data sets

Initial data (G, h, k, μ_0), $\mathfrak{g}$ the Lie algebra corresponding to G. Fix two-dimensional Abelian subalgebra $\mathfrak{g}_2$ of $\mathfrak{g}$ (Bianchi types other than VIII and IX), and orthonormal basis e_1, e_2, e_3 of $\mathfrak{g}$ such that e_2, e_3 span $\mathfrak{g}_2$.

The initial data is called *locally rotationally symmetric* if $\mathfrak{g}_2$ and basis can be chosen such that

- e_2 commutes with e_1 and e_3, i. e.$[e_2, e_1] = 0 = [e_2, e_3]$,
- the commutator $[e_1, e_3]$ is a multiple of e_2,
- the 2-tensor k_{ij} is diagonal, with $k_{11} = k_{33}$.

The initial data is called *of plane wave equilibrium type* if $\mathfrak{g}_2$ and basis can be chosen such that

$$\gamma_{1A}^{B} = -k_{AB},$$

where $[e_i, e_j] = \gamma_{ij}^{k} e_k$ and $A, B \in \{2, 3\}$.

The evolution equations

$$
\begin{aligned}
\Sigma_+' &= (q-2)\Sigma_+ - 2\tilde{N} \\
\tilde{\Sigma}' &= 2(q-2)\tilde{\Sigma} - 4\Sigma_+\tilde{A} - 4\Delta N_+ \\
\Delta' &= 2(q+\Sigma_+ - 1)\Delta + 2(\tilde{\Sigma} - \tilde{N})N_+ \\
\tilde{A}' &= 2(q+2\Sigma_+)\tilde{A} \\
N_+' &= (q+2\Sigma_+)N_+ + 6\Delta.
\end{aligned}
$$

with

$$
\begin{aligned}
q &= \frac{3}{2}(2-\gamma)(\Sigma_+^2 + \tilde{\Sigma}) + \frac{1}{2}(3\gamma-2)(1-\tilde{A}-\tilde{N}), \\
\tilde{N} &= \frac{1}{3}(N_+^2 - \kappa\tilde{A})
\end{aligned}
$$

and constraints $\tilde{\Sigma} \geq 0$, $\tilde{A} \geq 0$, $\tilde{N} \geq 0$ and

$$
\begin{aligned}
0 &= \tilde{\Sigma}\tilde{N} - \Delta^2 - \Sigma_+^2\tilde{A} \\
\Omega &= 1 - \Sigma_+^2 - \tilde{\Sigma} - \tilde{A} - \tilde{N} \geq 0.
\end{aligned}
$$

Invariant subsets

Notation	Restrictions
$B(\mathrm{VI}_\eta)$	$\kappa = 1/\eta < 0,\ \tilde{A} > 0$
$B^\pm(\mathrm{VII}_\eta)$	$\kappa = 1/\eta > 0,\ \tilde{A} > 0,\ N_+ > 0$ or $N_+ < 0$
$B^\pm(\mathrm{IV})$	$\kappa = 0,\ \tilde{A} > 0,\ N_+ > 0$ or $N_+ < 0$
$B(\mathrm{V})$	$\kappa = 0,\ \tilde{A} > 0,\ \Sigma_+ = \Delta = N_+ = 0$
$B^\pm(\mathrm{II})$	$\tilde{A} = 0,\ N_+ > 0$ or $N_+ < 0$
$B(\mathrm{I})$	$\tilde{A} = \Delta = N_+ = 0$

Table : Bianchi invariant sets with higher symmetry.

Notation	Class of models	Restrictions
$S^\pm(\mathrm{II})$	LRS Bianchi II	$\tilde{A} = 0,\ 3\Sigma_+^2 = \tilde{\Sigma},\ \Delta^2 = \Sigma_+^2 N_+^2$
$S(\mathrm{VI}_{-1})$	LRS Bianchi VI_{-1}	$\kappa = -1,\ \tilde{A} > 0,\ 3\Sigma_+^2 = \tilde{\Sigma},\ \Delta = \Sigma_+ N_+$
$S(\mathrm{VI}_\eta)$	Bianchi VI_η, $n^\alpha{}_\alpha = 0$	$\Delta = N_+ = 0,\ 3\Sigma_+^2 + \kappa\tilde{\Sigma} = 0,\ \tilde{A} > 0$
$S(\mathrm{V})$	Bianchi V FRW	$\kappa = 0,\ \Sigma_+ = \tilde{\Sigma} = \Delta = N_+ = 0$
$S^\pm(\mathrm{VII}_\eta)$	Bianchi VII_η FRW	$\Sigma_+ = \tilde{\Sigma} = \Delta = 0,\ \kappa\tilde{A} = N_+^2 > 0$