Spin-multipole effects in binary black holes & the test-body limit

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Capra 2016

see JV, D. Kunst, J. Steinhoff, T. Hinderer [arXiv:1601.07529]
and JV, J. Steinhoff [arXiv:1606.08832] (and references therein)

Overview

- The Mathisson-Papapetrou-Dixon (MPD) dynamics and effective action principles (surprising generality)

- Remarkable simplifications for the black hole case

- **To all orders in spin, at the leading PN orders,**

  for binary black holes, $(m_1,\, \boldsymbol{S}_1,\, m_2\,,\boldsymbol{S}_2)$

  —three ways to derive the same results:

(1) direct post-Newtonian (PN) calculation

(2) a **"test black hole"** with mass $\mu = \dfrac{m_1 m_2}{M}$ and spin $\boldsymbol{S}_{\text{test}} = \dfrac{m_2^2}{M^2}\boldsymbol{S}_1 + \dfrac{m_1^2}{M^2}\boldsymbol{S}_2$

in a **Kerr background** with mass $M = m_1 + m_2$ and spin $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$

(3) "deduced" in a certain way from a pole-dipole test body in Kerr

$$\left(\text{featuring Kerr with mass } M = m_1 + m_2 \text{ and spin } \boldsymbol{S}_0 = \frac{M}{m_1}\boldsymbol{S}_1 + \frac{M}{m_2}\boldsymbol{S}_2 \right)$$

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Effective worldline action principle

- Bodies $A = 1, 2$ with worldlines $x = z_A(\lambda)$ and metric $g_{\mu\nu}(x)$,

$$\mathcal{S}[z_A, g] = \frac{1}{16\pi}\int d^4x\sqrt{-g}R - \sum_A m_A \int d\lambda \sqrt{-g_{\mu\nu}(z_A)\dot{z}_A^\mu \dot{z}_A^\nu}$$

Formally, $\quad \Rightarrow \quad \ddot{z}^\mu = 0, \qquad G^{\mu\nu} = 8\pi T^{\mu\nu},$

$$T^{\mu\nu} = \sum_A m_A \int d\lambda\, u_A^\mu u_A^\nu \frac{\delta^4(x - z_A)}{\sqrt{-g}}, \qquad u_A^\mu = \frac{\dot{z}_A^\mu}{\sqrt{-\dot{z}_A^2}}$$

- Add rotational degrees of freedom (for each $A = 1, 2$)

—“body-fixed” tetrad $\Lambda_a{}^\mu(\lambda)$ along $x = z(\lambda)$ with $\Omega^{\mu\nu} = \Lambda_a{}^\mu \dfrac{D\Lambda^{a\nu}}{d\lambda}$,

$$\mathcal{S}_A = \int d\lambda\, \mathcal{L}_A\Big(\dot{z}^\mu,\, \Omega_{\mu\nu},\, g_{\mu\nu}(z),\, R_{\mu\nu\alpha\beta}(z),\, \nabla_\mu R_{\alpha\beta\gamma\delta}(z),\, \dots\Big)$$

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The MPD equations

- Define momentum $p_\mu = \dfrac{\partial \mathcal{L}}{\partial \dot{z}^\mu}$ and spin $S_{\mu\nu} = 2\dfrac{\partial \mathcal{L}}{\partial \Omega^{\mu\nu}}$,

  Action $\Rightarrow$ MPD equations: (force F^μ, torque $N^{\mu\nu}$)

$$\frac{Dp^\mu}{d\lambda} + \frac{1}{2}R^\mu{}_{\nu\alpha\beta}\dot{z}^\nu S^{\alpha\beta} = F^\mu, \qquad \frac{DS^{\mu\nu}}{d\lambda} - 2p^{[\mu}\dot{z}^{\nu]} = N^{\mu\nu}$$

  : transport eqs. for p^μ and $S^{\mu\nu}$ along any worldline.

- Add extra constraint, “spin supplementary condition” (SSC),

$$S_{\mu\nu}f^\nu = 0,$$

  (mass dipole vanishes in frame defined by timelike vector field f^μ) and MPD also determines evolution of worldline.

- $\Big($ Field eqs.?: $\quad \nabla_\nu p^\mu + \frac{1}{2}R^\mu{}_{\nu\alpha\beta}S^{\alpha\beta} = \ldots \quad , \quad \nabla_\alpha S^{\mu\nu} - 2p^{[\mu}\delta^{\nu]}{}_\alpha = \ldots \Big)$

Action for (quadrupolar) MPD

- Phase-space action, (α, β^μ : Lagrange multipliers)

$$\mathcal{S}_A[z,p,\Lambda,S] = \int d\lambda \left[p_\mu \dot{z}^\mu + \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} - \frac{\alpha}{2}\left(p^2 + \mathcal{M}^2\right) - \beta^\mu \mathcal{C}_\mu \right]$$

"dynamical mass" $\mathcal{M}^2(z,\hat{p},S)$ replaces Lagrangian $\mathcal{L}(z,\dot{z},\Omega)$

"spin-gauge constraint" : $\quad 0 = \mathcal{C}_\mu = S_{\mu\nu}\left(\hat{p}^\nu + {\Lambda_0}^\nu\right)$

- In general, $\Rightarrow$ MPD with

$$F_\mu = -\frac{\alpha}{2}\frac{\mathcal{D}\mathcal{M}^2}{\mathcal{D}z^\mu}, \quad N^{\mu\nu} = -\alpha\left(p^{[\mu}\frac{\partial\mathcal{M}^2}{\partial p_{\nu]}} + 2S^{[\mu}{}_\alpha \frac{\partial \mathcal{M}^2}{\partial S_{\nu]\alpha}}\right), \quad \alpha = \frac{p_\mu \dot{z}^\mu}{p^2}$$

- Define quadrupole, $\quad J^{\mu\nu\alpha\beta} = \frac{3p_\gamma \dot{z}^\gamma}{p^2}\frac{\partial\mathcal{M}^2}{\partial R_{\mu\nu\alpha\beta}}, \quad \left(\text{assume } \frac{\partial\mathcal{M}}{\partial\nabla R} = 0 = \ldots\right)$

$$\Rightarrow \qquad F_\mu = -\frac{1}{6}\nabla_\mu R_{\alpha\beta\gamma\delta}J^{\alpha\beta\gamma\delta} \quad , \qquad N^{\mu\nu} = \frac{4}{3}R^{[\mu}{}_{\alpha\beta\gamma}J^{\nu]\alpha\beta\gamma}$$

Quadrupolar couplings

- Define electric, magnetic parts of Weyl tensor,

$$\mathcal{E}_{\mu\nu} + i\mathcal{B}_{\mu\nu} = \left(C_{\mu\alpha\nu\beta} + i^*C_{\mu\alpha\nu\beta}\right)\hat{p}^\alpha\hat{p}^\beta,$$

mass dipole vector χ^μ, and Pauli-Lubanski spin vector s^μ,

$$\chi^\mu + is^\mu = -\left(S^{\mu\nu} + i^*S^{\mu\nu}\right)\hat{p}_\nu.$$

- Spin-induced and adiabatic tidal couplings:

$$\mathcal{M}^2 = m^2 - \kappa\mathcal{E}_{\mu\nu}s^\mu s^\nu - \lambda\frac{m}{2}\mathcal{E}_{\mu\nu}\mathcal{E}^{\mu\nu},$$

$$\left(\ \kappa_{\mathrm{BH}} = 1, \qquad \lambda_{\mathrm{BH}} = 0\ \right),$$

—valid for the "**covariant SSC**": $S_{\mu\nu}p^\nu = 0 \qquad (\chi^\mu = 0)$

- For a **generic SSC**, new kinematical terms: $(\lambda = 0)$

$$\mathcal{M}^2 = m^2 - \kappa\mathcal{E}_{\mu\nu}s^\mu s^\nu - 2\mathcal{B}_{\mu\nu}s^\mu\chi^\nu + \mathcal{E}_{\mu\nu}\chi^\mu\chi^\nu$$

Quadrupolar couplings for a black hole

- With $\kappa = 1$,

$$\begin{aligned}\mathcal{M}^2_{\rm BH} &= m^2 - \mathcal{E}_{\mu\nu} s^\mu s^\nu - 2\mathcal{B}_{\mu\nu} s^\mu \chi^\nu + \mathcal{E}_{\mu\nu}\chi^\mu\chi^\nu \\ &= m^2 + \frac{1}{2}(\mathcal{E}_{\mu\nu} + i\mathcal{B}_{\mu\nu})(\chi^\mu + i s^\mu)(\chi^\nu + i s^\nu) + c.c. \\ &= m^2 + \frac{1}{4} C_{\mu\nu\alpha\beta} S^{\mu\nu} S^{\alpha\beta}\end{aligned}$$

- Thus, for a BH (in vacuum),

$$J^{\mu\nu\alpha\beta} = \frac{3p\cdot\dot{z}}{4p^2}\Big(S^{\mu\nu}S^{\alpha\beta} - S^{[\mu\nu}S^{\alpha\beta]} - \text{traces}\Big).$$

- p–$\dot{z}$ relation for the cov. SSC, in general,

$$(-p\cdot\dot{z})p^\mu = (-p^2)\dot{z}^\mu - \frac{1}{2}S^{\mu\nu}R_{\nu\alpha\beta\gamma}\dot{z}^\alpha S^{\beta\gamma} + \frac{4}{3}R^{[\mu}{}_{\alpha\beta\gamma}J^{\nu]\alpha\beta\gamma}p_\nu + \mathcal{O}(S^3)$$

For a black hole, $$p^\mu = \frac{p\cdot\dot{z}}{p^2}\dot{z}^\mu + \mathcal{O}(S^3)$$

Quadrupolar couplings for a black hole

- With $\kappa = 1$,

$$\begin{aligned}\mathcal{M}^2_{\rm BH} &= m^2 - \mathcal{E}_{\mu\nu} s^\mu s^\nu - 2\mathcal{B}_{\mu\nu} s^\mu \chi^\nu + \mathcal{E}_{\mu\nu}\chi^\mu\chi^\nu \\ &= m^2 + \frac{1}{2}(\mathcal{E}_{\mu\nu} + i\mathcal{B}_{\mu\nu})(\chi^\mu + i s^\mu)(\chi^\nu + i s^\nu) + c.c. \\ &= m^2 + \frac{1}{4} C_{\mu\nu\alpha\beta} S^{\mu\nu} S^{\alpha\beta}\end{aligned}$$

- Thus, for a BH (in vacuum),

$$J^{\mu\nu\alpha\beta} = \frac{3p\cdot\dot{z}}{4p^2}\Big(S^{\mu\nu}S^{\alpha\beta} - S^{[\mu\nu}S^{\alpha\beta]} - \text{traces}\Big).$$

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For a black hole, $$p^\mu = \frac{p\cdot\dot{z}}{p^2}\dot{z}^\mu + \mathcal{O}(S^3)$$

LO-PN couplings for a BH to all orders in spin

- Rescale dipoles, $\xi^\mu + i\sigma^\mu = \dfrac{\chi^\mu + is^\mu}{m} = -\dfrac{1}{m}\big(S + i{}^*S\big)^{\mu\nu}\hat{p}_\nu$

  higher-order tidal tensors, $(\mathcal{P}_\mu^\nu = \delta_\mu^\nu + \hat{p}_\mu \hat{p}^\nu)$

$$\big(\mathcal{E} + i\mathcal{B}\big)_{\mu_1 \ldots \mu_\ell} = \mathcal{P}^{\nu_1}_{\mu_1} \ldots \mathcal{P}^{\nu_{\ell-2}}_{\mu_{\ell-2}} \nabla_{(\nu_1} \ldots \nabla_{\nu_{\ell-2}} \big(C + i{}^*C\big)_{\mu_{\ell-1}}{}^{\alpha}{}_{\mu_\ell)}{}^{\beta} \hat{p}_\alpha \hat{p}_\beta,$$

$$\begin{aligned}
\mathcal{M}^2 = m^2 + 2m^2\Big[& \\
& - \frac{1}{2!}\Big(\mathcal{E}_{\mu\nu}\sigma^\mu\sigma^\nu + 2\mathcal{B}_{\mu\nu}\sigma^\mu\xi^\nu - \mathcal{E}_{\mu\nu}\xi^\mu\xi^\nu\Big) \\
& + \frac{1}{3!}\Big(\mathcal{B}_{\mu\nu\alpha}\sigma^\mu\sigma^\nu\sigma^\alpha - 3\mathcal{E}_{\mu\nu\alpha}\sigma^\mu\sigma^\nu\xi^\alpha + \mathrm{NLO}\Big) \\
& + \frac{1}{4!}\Big(\mathcal{E}_{\mu\nu\alpha\beta}\sigma^\mu\sigma^\nu\sigma^\alpha\sigma^\beta + 4\mathcal{B}_{\mu\nu\alpha\beta}\sigma^\mu\sigma^\nu\sigma^\alpha\xi^\beta + \mathrm{NLO}\Big) \\
& - \frac{1}{5!}\Big(\mathcal{B}_{\mu\nu\alpha\beta\gamma}\sigma^\mu\sigma^\nu\sigma^\alpha\sigma^\beta\sigma^\gamma - 5\mathcal{E}_{\mu\nu\alpha\beta\gamma}\sigma^\mu\sigma^\nu\sigma^\alpha\sigma^\beta\xi^\gamma + \mathrm{NLO}\Big) \\
& - \ldots \Big]
\end{aligned}$$

The PN and spin expansions (by PN order)

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
spin^0	N	1PN		2PN		3PN		4PN				
spin^1			LO SO		NLO SO		NNLO SO					
spin^2				LO S^2		NLO S^2		NNLO S^2				
spin^3							LO S^3		NLO S^3			
spin^4								LO S^4		NLO S^4		
spin^5											LO S^5	
spin^6												LO S^6

“nPN” : no-spin / point-mass, “SO” : spin-orbit / linear-in-spin, . . .

“LO” : leading-(PN-)order, “NLO” : next-to-leading-order, . . .

The PN-spin expansion (rearranged)

PN order			1.5		2.5		3.5		4.5		5.5	
0		1		2		3		4		5		6
N		1PN		2PN		3PN		4PN				
			LO SO		NLO SO		NNLO SO					
				LO S^2		NLO S^2		NNLO S^2				
							LO S^3		NLO S^3			
								LO S^4		NLO S^4		
											LO S^5	
												LO S^6

LO even		NLO even		...				
	LO odd		NLO odd		...			
N		1PN		2PN		3PN		4PN
	LO SO		NLO SO		NNLO SO		NNNLO SO	
LO S^2		NLO S^2		NNLO S^2		NNNLO S^2		
	LO S^3		NLO S^3		NNLO S^3			
LO S^4		NLO S^4		NNLO S^4				
	LO S^5		NLO S^5					
LO S^6		NLO S^6						
	LO S^7					nPN		(n+1)PN
LO S^8					(n+0.5)PN			
	...							

Hamiltonian $H = H_{\rm N} + H_{\rm 1PN} + \ldots$

PN counting assumes large spins $S \sim Gm^2/c$.

(for arbitrary-mass-ratio binaries with spin-induced body multipoles)

The PN-spin expansion

Red text: not (fully) known

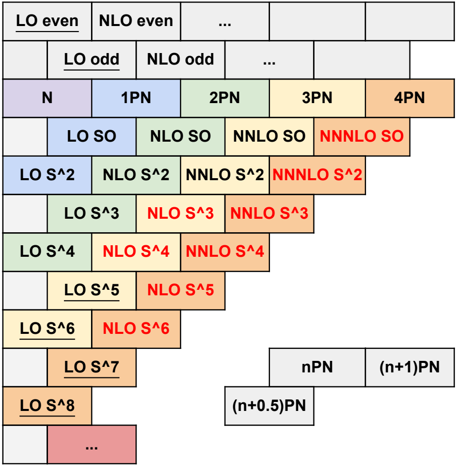
Black text: fully calculated,

and confirmed, all except for:

NNLO-S^2

4PN

LO-S^n with $n \geq 5$



PN compact binaries

Describe binary of compact objects, bodies $A = 1, 2$ in terms of

- worldlines $\boldsymbol{x} = \boldsymbol{z}_A(t)$ in PN coordinates $x^\mu = (t, x^i) = (t, \boldsymbol{x})$,

  relative position $\boldsymbol{R} = \boldsymbol{z}_2 - \boldsymbol{z}_1$, distance $R = |\boldsymbol{R}|$,

- masses m_A $(M = m_1 + m_2,\ \ \mu = m_1 m_2 / M,\ \ \nu = \mu/M)$,

  take $m_1 \geq m_2$, "test-body limit" : $m_2 \to 0$,

- spin vectors $\boldsymbol{S}_A = S^i_A$, rescaled spins $\boldsymbol{a}_A = \boldsymbol{S}_A / m_A c$,

- assume only spin-induced multipole moments, $H(\boldsymbol{R}, \boldsymbol{P}, \boldsymbol{S}_1, \boldsymbol{S}_2)$,

$$\dot{R}^i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial R^i}, \quad \dot{S}^i_A = \epsilon^{ij}{}_k \frac{\partial H}{\partial S^j_A} S^k_A, \tag{1}$$

rescale momenta: $\bar{H} = \dfrac{H}{\mu}$, $\bar{\boldsymbol{P}} = \dfrac{\boldsymbol{P}}{\mu}$, $\bar{\boldsymbol{L}} = \dfrac{\boldsymbol{L}}{\mu} = \boldsymbol{R} \times \bar{\boldsymbol{P}}$,

Leading-order Hamiltonians

- Newtonian point-mass:

$$\bar{H}_{\rm N} = \frac{\bar{\boldsymbol{P}}^2}{2} - \frac{M}{R},$$

- 1PN point-mass:

$$\bar{H}_{\rm 1PN} = (-1+3\nu)\frac{\bar{\boldsymbol{P}}^4}{8} + (-3-2\nu)\frac{M\bar{\boldsymbol{P}}^2}{2R} + (0+\nu)\frac{M\bar{\boldsymbol{L}}^2}{2R^3} + (1+0\nu)\frac{M^2}{2R^2}.$$

- Leading-order spin-orbit: (spin $\boldsymbol{S} = m\boldsymbol{a}$)

$$\bar{H}_{\text{LO-S}^1} = \left(2m_1 + \frac{3}{2}m_2\right)\frac{\bar{\boldsymbol{L}}\cdot\boldsymbol{a}_1}{R^3} + \left(\frac{3}{2}m_1 + 2m_2\right)\frac{\bar{\boldsymbol{L}}\cdot\boldsymbol{a}_2}{R^3}.$$

Leading-order spin-orbit

$$\begin{aligned}
\bar{H}_{\text{LO-S}^1} &= \left(2m_1 + \frac{3}{2}m_2\right)\frac{\bar{\boldsymbol{L}}\cdot\boldsymbol{a}_1}{R^3} + \left(\frac{3}{2}m_1 + 2m_2\right)\frac{\bar{\boldsymbol{L}}\cdot\boldsymbol{a}_2}{R^3} \\
&= \bar{\boldsymbol{L}}\cdot\left(2\boldsymbol{a} + \frac{3}{2}\boldsymbol{\sigma}\right)\frac{M}{R^3} \\
&= -\bar{\boldsymbol{P}}\times\left(2\boldsymbol{a} + \frac{3}{2}\boldsymbol{\sigma}\right)\cdot\boldsymbol{\partial}\,\frac{M}{R},
\end{aligned}$$

- Spin map:

$$\begin{aligned}
\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 &= m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2 = M\boldsymbol{a}, \\
\frac{\boldsymbol{S}_{\text{test}}}{\nu} = \boldsymbol{S}^* = \frac{m_1}{m_2}\boldsymbol{S}_2 + \frac{m_2}{m_1}\boldsymbol{S}_1 &= m_1\boldsymbol{a}_2 + m_2\boldsymbol{a}_1 = M\boldsymbol{\sigma},
\end{aligned}$$

- Equivalent to the motion of a test body:

$$\bar{H}_{\text{LO-S}^1}(m_1, \boldsymbol{a}_1, m_2, \boldsymbol{a}_2) = \bar{H}^{\text{test}}_{\text{LO-S}^1}(M, \boldsymbol{a}, \mu, \boldsymbol{\sigma})$$

Leading-order spin-squared

$$\bar{H}_{\text{LO-S}^2} = \frac{1}{2}\Big(\kappa_1 a_1^i a_1^j + 2a_1^i a_2^j + \kappa_2 a_2^i a_2^j\Big)\partial_i\partial_j \frac{M}{R},$$

- κ : response coefficient for spin-induced quadrupole : $\kappa_{\text{BH}} = 1$

$$\begin{aligned}
\bar{H}^{\text{BBH}}_{\text{LO-S}^2}(m_1, \boldsymbol{a}_1, m_2, \boldsymbol{a}_2) &= \frac{1}{2}\big(\boldsymbol{a}_1 + \boldsymbol{a}_2\big)^i \big(\boldsymbol{a}_1 + \boldsymbol{a}_2\big)^j \partial_i\partial_j \frac{M}{R} \\
= \bar{H}^{\text{BBH,test}}_{\text{LO-S}^2}(M, \boldsymbol{a}, \mu, \boldsymbol{\sigma}) &= \frac{1}{2}\big((\boldsymbol{a} + \boldsymbol{\sigma})\cdot\boldsymbol{\partial}\big)^2 \frac{M}{R} \\
= \bar{H}^{\text{BBH,test}}_{\text{LO-S}^2}(M, \boldsymbol{a}_0, \mu, 0) &= \frac{1}{2}(\boldsymbol{a}_0\cdot\boldsymbol{\partial})^2 \frac{M}{R}
\end{aligned}$$

where

$$\boldsymbol{a}_0 = \boldsymbol{a}_1 + \boldsymbol{a}_2 = \boldsymbol{a} + \boldsymbol{\sigma} = \frac{\boldsymbol{S} + \boldsymbol{S}^*}{M} = \frac{\boldsymbol{S}_0}{M}$$

Through S^4, at the leading PN orders, for BBHs

- Even part:

$$\bar{H}^{\rm BBH}_{\rm LO,even} = \frac{\bar{\boldsymbol{P}}^2}{2} - \frac{M}{R} + \frac{1}{2!}(\boldsymbol{a}_0 \cdot \boldsymbol{\partial})^2 \frac{M}{R} - \frac{1}{4!}(\boldsymbol{a}_0 \cdot \boldsymbol{\partial})^4 \frac{M}{R} + \mathcal{O}(S^6),$$

- Odd part:

$$\bar{H}^{\rm BBH}_{\rm LO,odd} = -\frac{1}{1!}\bar{\boldsymbol{P}} \times \left(2\boldsymbol{a} + \frac{3}{2}\boldsymbol{\sigma}\right) \cdot \boldsymbol{\partial}\, \frac{M}{R} + \frac{1}{3!}\bar{\boldsymbol{P}} \times \left(2\boldsymbol{a} + \frac{1}{2}\boldsymbol{\sigma}\right) \cdot \boldsymbol{\partial}\, (\boldsymbol{a}_0 \cdot \boldsymbol{\partial})^2 \frac{M}{R} + \mathcal{O}(S^5).$$

Arbitrary-mass-ratio results from the test-body limit

- Even and odd parts, from a "test black hole"—
  —with all the multipoles—in Kerr:

$$\bar{H}^{\mathrm{BBH}}_{\mathrm{LO}}(m_1, \boldsymbol{a}_1, m_2, \boldsymbol{a}_2) = \bar{H}^{\mathrm{BBH,test}}_{\mathrm{LO}}(M, \boldsymbol{a}, \mu, \boldsymbol{\sigma}),$$

- The even part, from geodesics in Kerr:

$$\bar{H}^{\mathrm{BBH}}_{\mathrm{LO,even}}(m_1, \boldsymbol{a}_1, m_2, \boldsymbol{a}_2) = \bar{H}^{\mathrm{BBH,test}}_{\mathrm{LO,even}}(M, \boldsymbol{a}_0, \mu, 0),$$

- Even and odd parts "deduced" in a certain way
  from a pole-dipole test body in Kerr

To all orders in spin, even part

$$\bar{H}^{\mathrm{BBH}}_{\mathrm{LO,even}} - \frac{\bar{\boldsymbol{P}}^2}{2} = -\frac{M}{R} + \frac{1}{2!}(\boldsymbol{a}_0 \cdot \boldsymbol{\partial})^2 \frac{M}{R} - \frac{1}{4!}(\boldsymbol{a}_0 \cdot \boldsymbol{\partial})^4 \frac{M}{R} + \dots$$

$$= -\sum_{\ell}^{\mathrm{even}} \frac{i^\ell}{\ell!}(\boldsymbol{a}_0 \cdot \boldsymbol{\partial})^\ell \frac{M}{R}$$

$$= -\cos(\boldsymbol{a}_0 \cdot \boldsymbol{\partial}) \frac{M}{R}$$

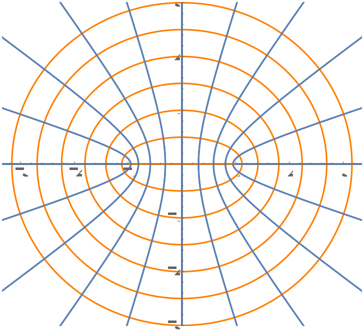
$$= -\exp(i\boldsymbol{a} \cdot \boldsymbol{\partial}) \frac{M/2}{R}$$

$$= -\left(\frac{M/2}{|\boldsymbol{R} + i\boldsymbol{a}_0|} + c.c.\right)$$

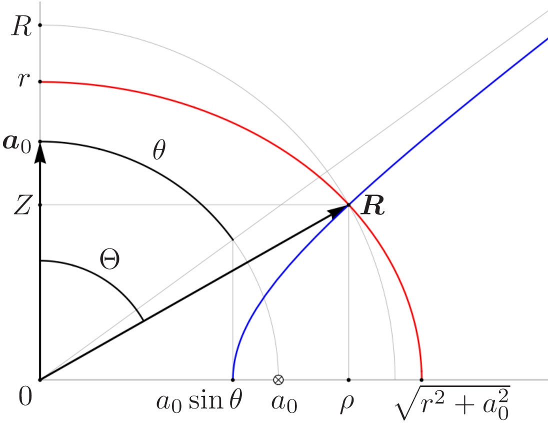
$$= -\frac{Mr}{r^2 + a_0^2 \cos^2\theta}$$

The **oblate spheroidal (Kerr-Schild) geometry** naturally emerges, with a ring-disk singularity of radius $a_0 = |\boldsymbol{a}_0|$.

Oblate spheroidal geometry



const. r – ellipsoid
const. θ – hyperboloid



coordinates:

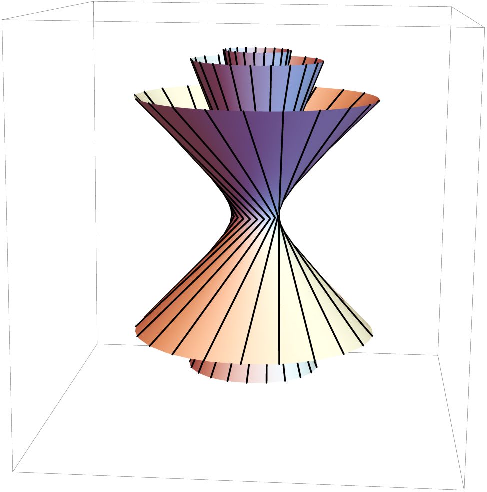
$$\text{cylindrical } (\rho, \Phi, Z), \quad X = \rho\cos\Phi, \quad Y = \rho\sin\Phi,$$

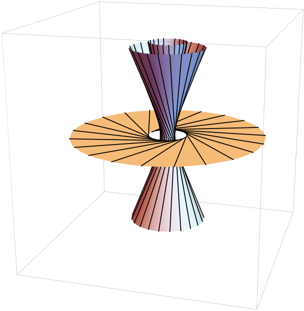
$$\text{spherical } (R, \Theta, \Phi), \quad \rho = R\sin\Theta, \quad Z = R\cos\Theta,$$

$$\text{spheroidal } (r, \theta, \Phi), \quad \rho = \sqrt{r^2 + a_0^2}\,\sin\theta, \quad Z = r\cos\theta.$$

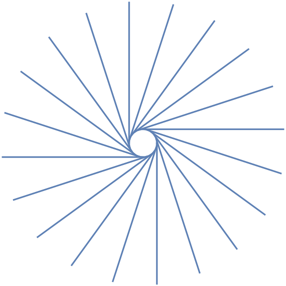
Oblate spheroidal geometry

the ingoing principal null congruence





equatorial plane →



To all orders in spin, odd part

$$\bar{H}^{\rm BBH}_{\rm LO,odd} = -\frac{1}{1!}\bar{\boldsymbol{P}}\times\left(2\boldsymbol{a}+\frac{3}{2}\boldsymbol{\sigma}\right)\cdot\boldsymbol{\partial}\,\frac{M}{R}$$
$$+\frac{1}{3!}\bar{\boldsymbol{P}}\times\left(2\boldsymbol{a}+\frac{1}{2}\boldsymbol{\sigma}\right)\cdot\boldsymbol{\partial}\,(\boldsymbol{a}_0\cdot\boldsymbol{\partial})^2\,\frac{M}{R}+\ldots$$

$$=\sum_{\ell}^{\rm odd}\frac{i^{\ell-1}}{\ell!}\bar{\boldsymbol{P}}\times\left(-2\boldsymbol{a}+\frac{\ell-4}{2}\boldsymbol{\sigma}\right)\cdot\boldsymbol{\partial}\,(\boldsymbol{a}_0\cdot\boldsymbol{\partial})^{\ell-1}\,\frac{M}{R}$$

$$=\left[-\,2\,\bar{\boldsymbol{P}}\times\boldsymbol{a}_0\cdot\boldsymbol{\partial}\,\frac{\sin(\boldsymbol{a}_0\cdot\boldsymbol{\partial})}{\boldsymbol{a}_0\cdot\boldsymbol{\partial}}+\frac{1}{2}\,\bar{\boldsymbol{P}}\times\boldsymbol{\sigma}\cdot\boldsymbol{\partial}\,\cos(\boldsymbol{a}_0\cdot\boldsymbol{\partial})\right]\frac{M}{R}$$

$$=\frac{Mr}{r^2+a_0^2\cos^2\theta}\,\frac{2\boldsymbol{R}\times\bar{\boldsymbol{P}}\cdot\boldsymbol{a}_0}{r^2+a_0^2}-\frac{M}{4}\bar{\boldsymbol{P}}\times\boldsymbol{\sigma}\cdot\left(\frac{\boldsymbol{R}+i\boldsymbol{a}_0}{(r+ia_0\cos\theta)^3}+c.c.\right)$$

Summary

- **To all orders in spin, at the leading PN orders,**

  for binary black holes, $(m_1,\, \boldsymbol{S}_1,\, m_2\,,\boldsymbol{S}_2)$

  —three ways to derive the same results:

(1) direct post-Newtonian (PN) calculation

(2) a **"test black hole"** with mass $\mu = \dfrac{m_1 m_2}{M}$ and spin $\boldsymbol{S}_{\text{test}} = \dfrac{m_2^2}{M^2}\boldsymbol{S}_1 + \dfrac{m_1^2}{M^2}\boldsymbol{S}_2$

in a **Kerr background** with mass $M = m_1 + m_2$ and spin $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$

(3) "deduced" in a certain way from a pole-dipole test body in Kerr

$\left(\text{featuring Kerr with mass } M = m_1 + m_2 \text{ and spin } \boldsymbol{S}_0 = \dfrac{M}{m_1}\boldsymbol{S}_1 + \dfrac{M}{m_2}\boldsymbol{S}_2\right)$

- Oblate spheroidal (effective) Kerr-Schild geometry naturally emerges from exact resummation of the LO conservative dynamics.