Parametrized theories: Making EM even “gaugier”

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Joint work with F. Barbero (CSIC) and Eduardo J.S. Villaseñor (UC3M)

21st International Conference on General Relativity and Gravitation

July 2016 - New York

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$(M \cong \Sigma \times I, g)$ being $\begin{array}{l} \Sigma \ \ 3\text{-manifold} \\ I = [t_0, t_1] \end{array}$

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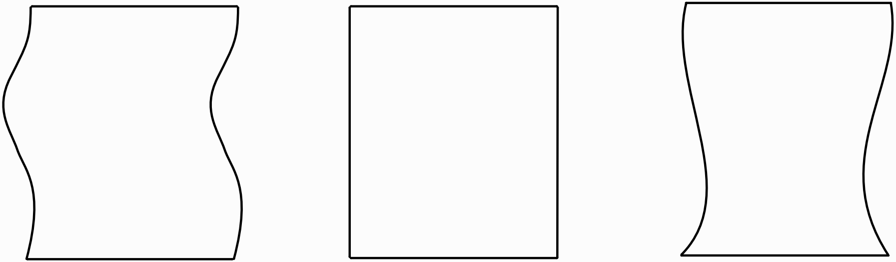

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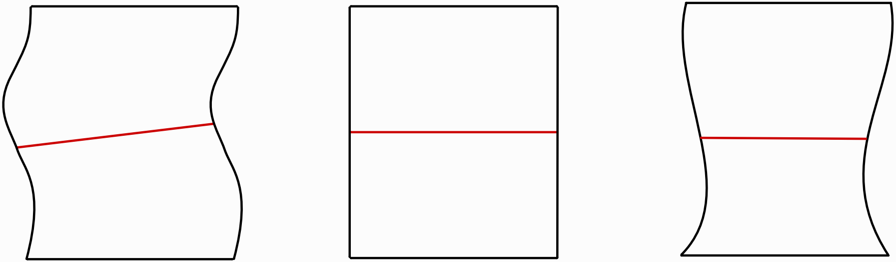

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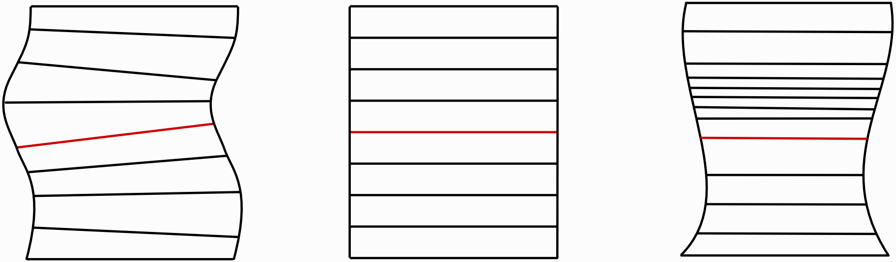

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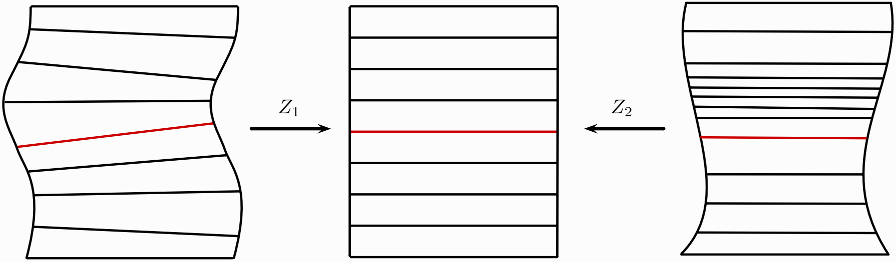

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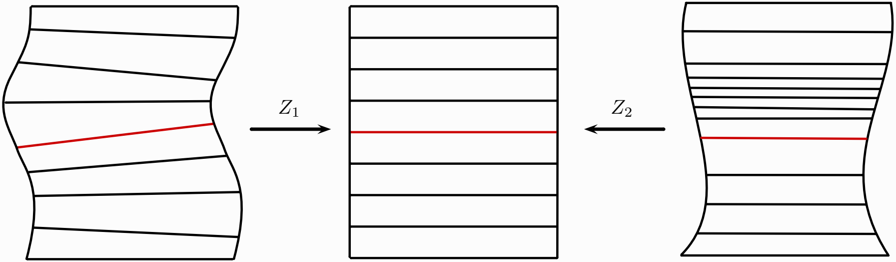

$$Diff^+_{sp}(M) := \left\{ Z : M \to M \text{ diff. } / \begin{array}{ll} TZ.\partial_t & \text{future timelike} \\ Z(\Sigma \times \{t\}) & \text{spacelike} \end{array} \right\}$$

Our model

Parametrized EM Field Action

$$S_{EM} : \Omega^1(M) \longrightarrow \mathbb{R}$$

$$S_{EM}[A] = \int_M (\mathrm{d}A) \wedge (\star_g \mathrm{d}A)$$

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Properties

- If (A, Z) is a solution, then so is $(Y^*A, Z \circ Y)$.

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Important things to retain

- L^P_{EM} homogeneous of degree 1, hence $H^P_{EM} = 0$.

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- In fact, this is true for any parametrized theory.

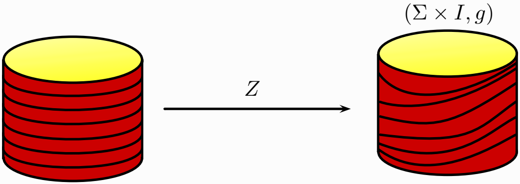
The geometric arena

Configuration space

$\Omega^1(M) \times Diff^+_{\mathrm{sp}}(M)$

$q^{(4)} : \Sigma \times I \to T^*(\Sigma \times I)$

$Z : \Sigma \times I \to \Sigma \times I$



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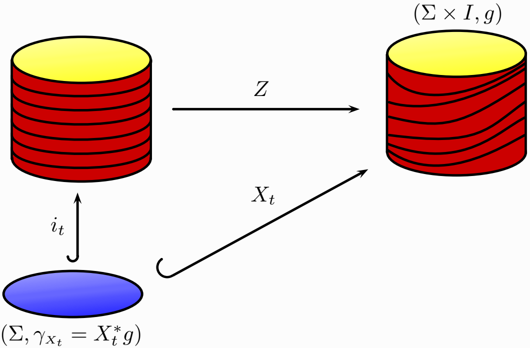

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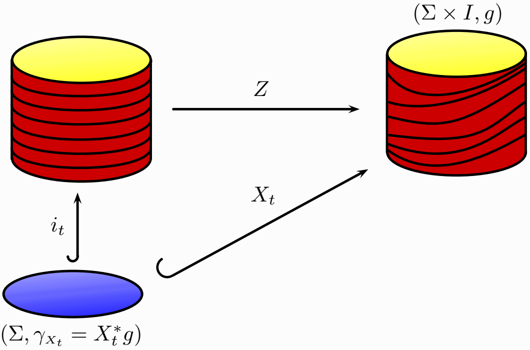

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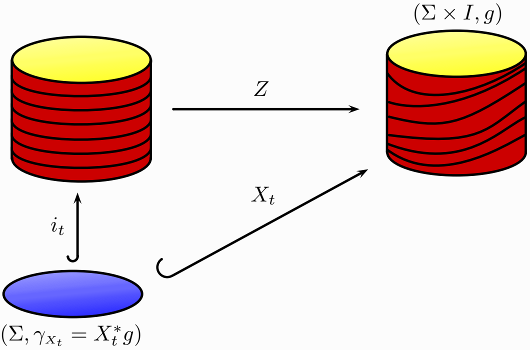

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$$\Omega^1(M) \times Diff^+_{\rm sp}(M) \qquad Q_{EM} = C^\infty(\Sigma) \times \Omega^1(\Sigma) \times \mathrm{Emb}(\Sigma, M)$$

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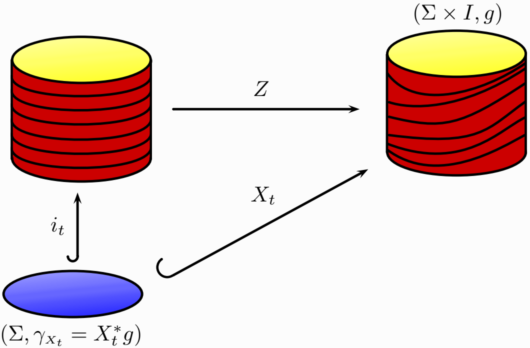

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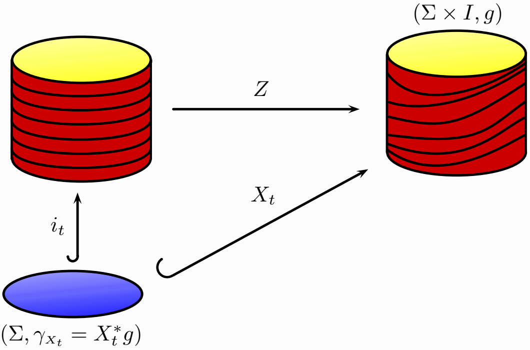

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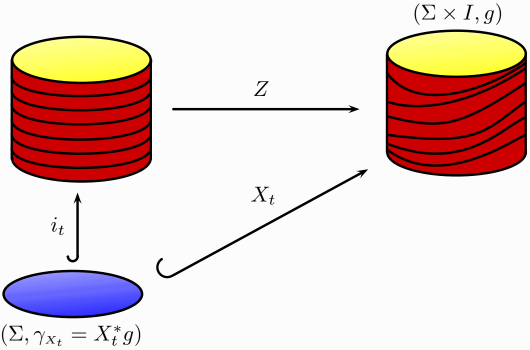

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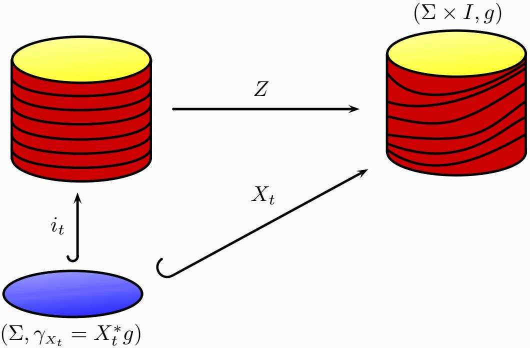

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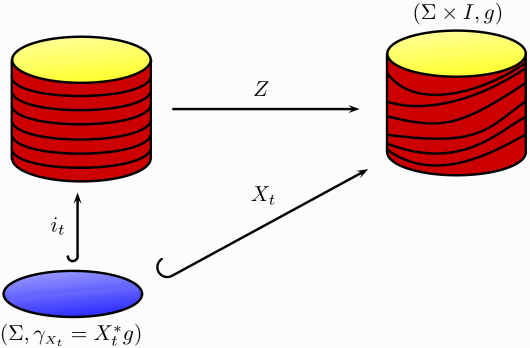

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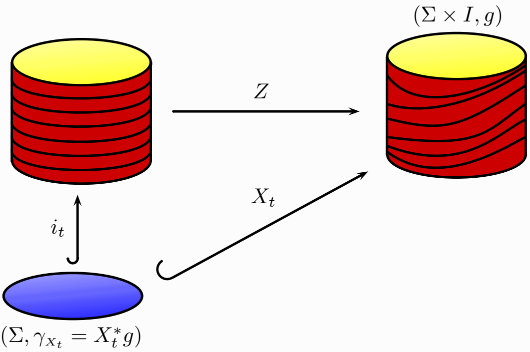

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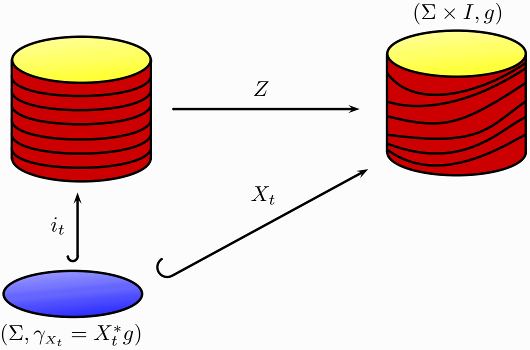

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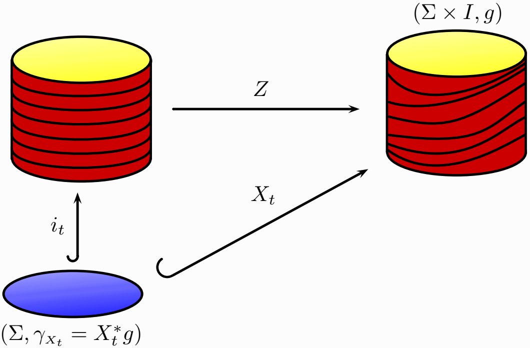

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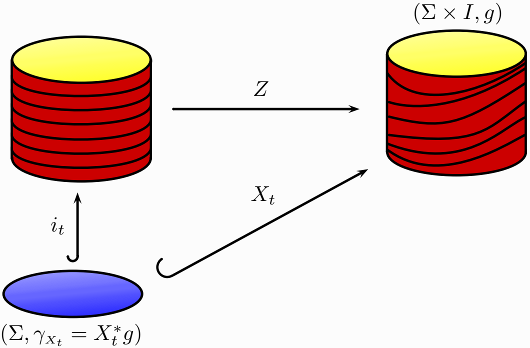
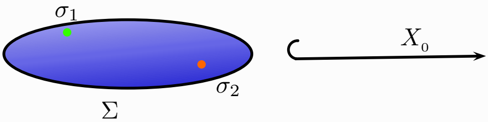


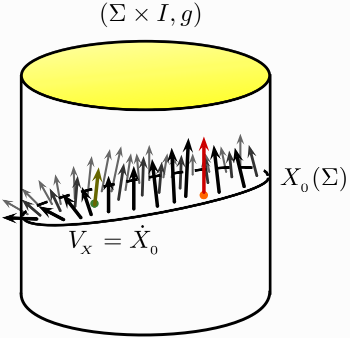
$T\mathrm{Emb}(\Sigma, M)$

$$\begin{tikzcd} & TM \arrow[d] \\ \Sigma \arrow[ur, "V_X"] \arrow[r, hook, "X"'] & M \end{tikzcd}$$

$$V_X \in T_X \mathrm{Emb}(\Sigma, M)$$

$$\{X_\lambda\} \quad / \quad \left| \begin{array}{l} X_0 = X \\ \dot{X}_0 = V_X \end{array} \right.$$





Back to the Hamiltonian

Hamiltonian vector field

Symplectic form

$$\left(FL^P_{EM}(\mathcal{D}), j^*\Omega\right) \overset{j}{\hookrightarrow} \left(T^*Q_{EM}, \Omega\right)$$

If $Z \in \mathfrak{X}(FL^P_{EM}(\mathcal{D}))$ then $Z = \left(q_\perp, q, X, p; Z_{\mathsf{q}_\perp}, Z_{\mathsf{q}}, \vec{Z}_{\mathsf{X}}, Z_{\mathsf{p}}\right)$

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- $Z^\perp_{\mathsf{X}}(\nabla_b p^b) = 0$
- Z^T_{X} arbitrary
- $\left(Z_{\mathsf{q}_\perp} - \mathcal{L}_{Z^T_{\mathsf{X}}} q_\perp + q_a \nabla^a Z^\perp_{\mathsf{X}}\right)\nabla_b p^b = 0$

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Thanks for your attention

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