Shear-free surfaces and distinguished dynamical horizons

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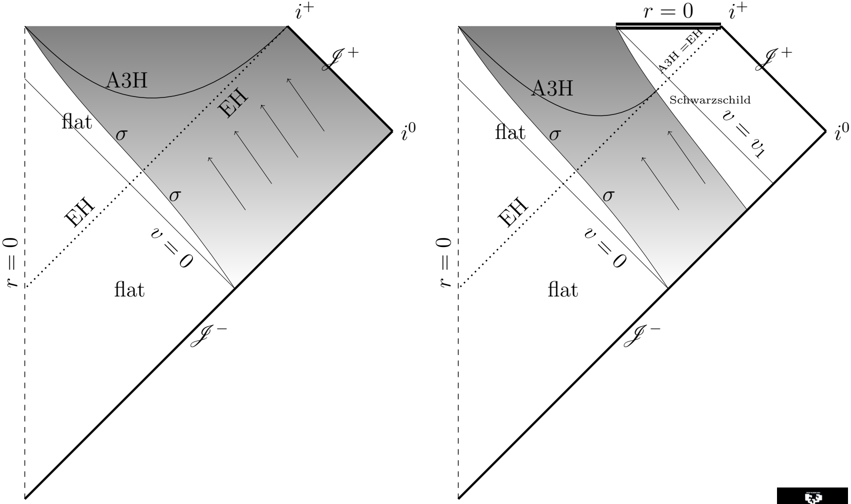
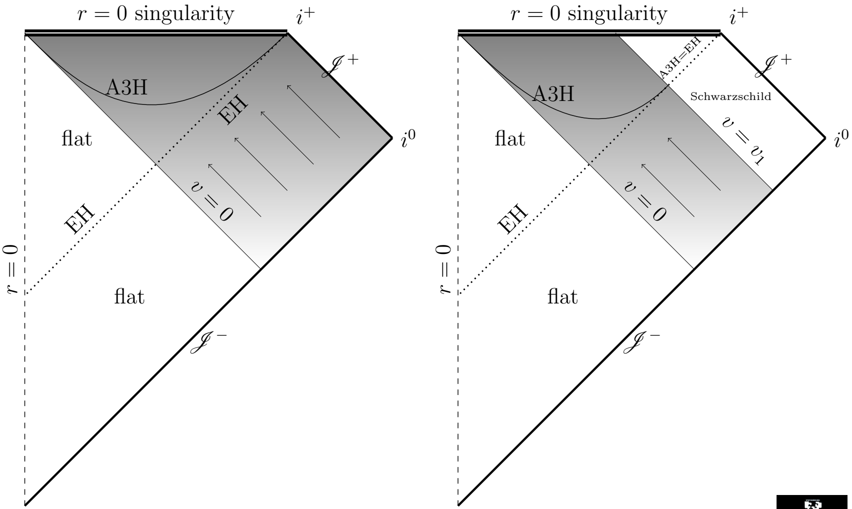
Outline

Creation of a Black Hole (BH)



Observe the central line



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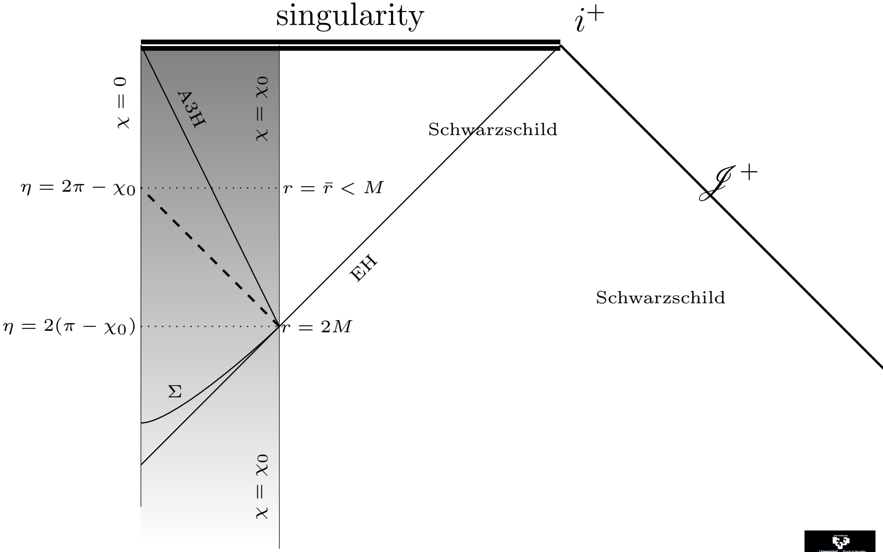
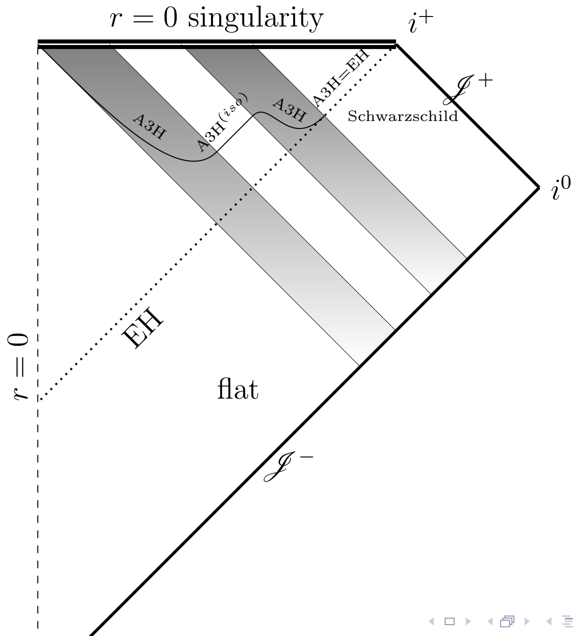
- If the MTT is null with null generator along the direction of vanishing expansion, they are called *non-expanding horizons* (NEH)
- NEHs include the so-called *isolated horizons* as well as the *Killing horizons*
- Of course, the MTT of a realistic dynamical black hole can be composed of several portions of each of these kinds!

Example: The Oppenheimer-Snyder Black Hole



Case with a portion of Isolated Horizon A3H$^{(iso)}$



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- Therefore, it would be very welcomed to have a physically sound criterion selecting a preferred MTT.
- So far, all tries have failed or been inconclusive (evolution maximizing entropy (Gourgoulhon & Jaramillo, Phys.Rev. D74 (2006) 087502), cores (Bengtsson & Senovilla, Phys.Rev.D83 (2011) 044012) ,...)

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- The shape tensor (or second fundamental form vector) is

$$\vec{K}_{AB} = -K_{AB}(\vec{k}^{-})\, \vec{k}^{+} - K_{AB}(\vec{k}^{+})\, \vec{k}^{-}$$

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- Observe that $K_{AB}(\vec{k}^{\pm})$ are affected by the gauge freedom above, but not so $\vec{K}_{AB}$.

Mean curvature vector. Null expansions

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Future-trapped surfaces are characterized by the causal orientation of its mean curvature vector:

$\vec{H}$	Expansions	Type of surface
future timelike	$\theta^+ < 0, \theta^- < 0$	f-trapped
future causal	$\theta^+ \leq 0, \theta^- \leq 0$	weakly f-trapped
future null	$\theta^+ = 0, \theta^- \leq 0$	marginally f-trapped
null	$\theta^+ = 0$	marginally outer trapped
zero	$\theta^+ = \theta^- = 0$	stationary or minimal

Shear scalars along null normals

- The classical shear scalars along $\vec{k}^{\pm}$ are given, at the surface S, by

$$(\sigma^{\pm})^2 := \left(K_{AB}(\vec{k}^{\pm}) - \frac{1}{2}\theta^{\pm}\gamma_{AB}\right)\left(K^{AB}(\vec{k}^{\pm}) - \frac{1}{2}\theta^{\pm}\gamma^{AB}\right)$$

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- Equivalently, this states that the $(+)$ null second fundamental form is proportional to the first fundamental form. In mathematics this property characterizes surfaces umbilical along $\vec{k}^{+}$.

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- **Remark:** Observe that $\vec{n}$ can have any causal character (and even change causal character from point to point).

Characterization of shear-free surfaces

Theorem

The necessary and sufficient condition for a surface S to be shear-free along a normal direction is that the two null second fundamental forms commute:

$$K^{A}{}_{C}(\vec{k}^{+})K^{C}{}_{B}(\vec{k}^{-}) = K^{A}{}_{C}(\vec{k}^{-})K^{C}{}_{B}(\vec{k}^{+})$$

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- (unless this object vanishes, which corresponds to totally umbilical surfaces, i.e., surfaces that are shear-free along *all* normal directions).

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- Notice that the shear-free direction is given by

$$\vec{n} = \star\vec{G} := \sigma^{+}\vec{k}^{-} - \sigma^{-}\vec{k}^{+}$$

(if it exists!)

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- From now on I will use ESF-MTT

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$$ds^2 = -dt^2 + F^2 dr^2 + G^2 d\Omega_\epsilon^2$$

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- The surfaces defined by constant values of t and r are totally umbilical again. There is then a hypersurface, defined by

$$FG_{,t} = G_r$$

which an ESF-MTT. This coincides with the canonical horizon usually studied in these metics.

Supporting facts (continued)

- Consider now **Robinson-Trautman** metrics

$$ds^2 = -F du^2 - 2du dr + \frac{r^2}{P^2} d\Omega_\epsilon^2$$

with $d\Omega_\epsilon^2$ as above, F is an arbitrary function and P is a function independent of r.

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- (Even more so in the previous examples...)

More supporting facts: Kerr

- As a non-trivial example consider the Kerr spacetime, given in Kerr coordinates by

$$
-\left(1-\frac{2mr}{\rho^2}\right)dv^2+2dvdr+\rho^2d\theta^2-\frac{4amr\sin^2\theta}{\rho^2}dvd\varphi
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- This spacetime is Petrov type D, thus with two *shear-free* principal null directions... However, they are <u>not</u> irrotational, and thus they are not orthogonal to surfaces!

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- As a non-trivial example consider the Kerr spacetime, given in Kerr coordinates by

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$$-2a\sin^2\theta drd\varphi+\frac{(r^2+a^2)^2-a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\varphi^2$$

- The notation is standard, with $\rho^2=r^2+a^2\cos^2\theta$, $\Delta=r^2-2mr+a^2$ and m and a two constants.
- This spacetime is Petrov type D, thus with two *shear-free* principal null directions... However, they are <u>not</u> irrotational, and thus they are not orthogonal to surfaces!
- We consider the preferred surfaces defined by constant values of v and r (these are topological spheres). When are they shear-free?

Shear-free surfaces in Kerr spacetime

Result (Shear-free surfaces in Kerr)

In the Kerr spacetime with $a \neq 0$, the only shear-free surfaces defined by constant values of v and r are those sitting on either

1. *the (timelike) hypersurface $r = 0$ or*
2. *the (null) hypersurface $\Delta = 0$ —these exist only when $m \geq |a|$*

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1. In the first case the surfaces are non compact (topological disks), they are untrapped, and they happen to be locally flat. (No interest here)
2. The two connected components of the second case define the Cauchy horizon ($r = r_-$) and the Event horizon ($r = r_+$) of the Kerr black hole, and the foliating surfaces are marginally trapped.

Non-expanding horizons

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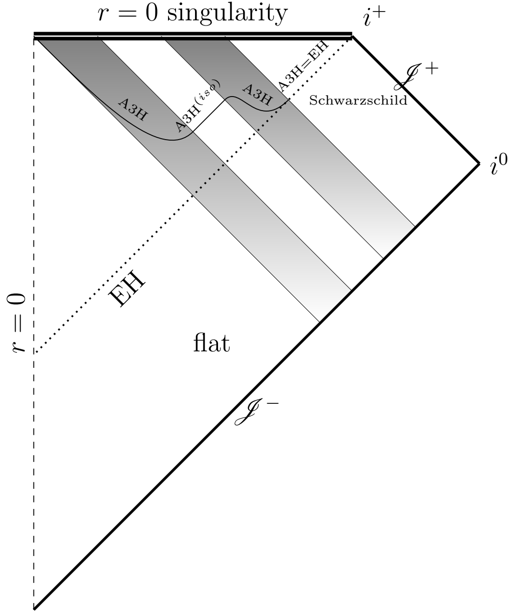
- Non-expanding horizons (including isolated and Killing horizons) are null hypersurfaces with a shear-free and expansion-free null generator
- Any cut of such a null hypersurface will thus be a shear-free surface
- Thus, all NEH are in particular ESF-MTT

Recall



$r = 0$ singularity
i^+
$\mathscr{I}^+$
Schwarzschild
A3H=EH
A3H
A3H$^{(iso)}$
A3H
i^0
EH
$r = 0$
flat
$\mathscr{I}^-$

A tougher test: Gowdy spacetimes

- Consider Gowdy-like spacetimes, given in local coordinates by

$$
\begin{aligned}
L^{-2}ds^2 = e^{2a}\left(-dt^2 + \mathrm{d}\psi^2\right) + 2\left(M d\theta + N d\varphi\right) dt \\
+G\left[e^{p}\left(d\theta + W d\varphi\right)^2 + e^{-p} d\varphi^2\right]
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- Here, the hypersurfaces $t =$const. are assumed to be closed, and therefore the coordinates ψ, θ, φ are angular coordinates. The level hypersurfaces can describe several topologies, such as $S^1 \times S^1 \times S^1$, or $S^2 \times S^1$, or S^3.

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- ∂_θ and ∂_φ are commuting Killing vectors, so that the functions a, M, N, G, p depend only on t and ψ.
- For simplicity, we concentrate on the surfaces of transitivity of the G_2 group of motions, given by constant values of t and ψ.

ESF-MTTs in Gowdy spacetimes

- The surfaces of transitivity are marginally trapped if and only if

$$e^{4a}G^2G_{,t}^2 + \det(g)G_{,\psi}^2 = 0$$

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- On the other hand, the condition for a MTT depends crucially on G —and actually only on G if the group is orthogonally transitive ($M = N = 0$).
- The existence of ESF-MTTs is easily checked: one trivial example is the case with orthogonal Killing vectors, $W = 0$.

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- Then one can use variation methods: deform the surface along normal directions and see if one can keep the shear-free property along (at least) one of those MTTs
- The second level is much more difficult and ambitious –and yet unexplored.
- Starting from a MTS S, it consists in trying to prove that at least some of the MTTs containing S will eventually develop a shear-free MTS.

Deforming shear-free MTS

- Take a marginally trapped S (so that $\theta^+ = 0$) and assume that it is shear-free. As explained above, this means that

$$\left[\boldsymbol{K}(\vec{k}^+), \boldsymbol{K}(\vec{k}^-)\right] = \mathbf{0}$$

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$$\delta_{\vec{\xi}} K_{AB}(\vec{k}^\pm) \mp \kappa_{\vec{\xi}} K_{AB}(\vec{k}^\pm) = -\overline{\nabla}_A \overline{\nabla}_B \xi^\pm \pm 2 s_{(A}\overline{\nabla}_{B)}\xi^\pm$$
$$-\xi^\mp \left(K_{AC}(\vec{k}^\pm) K_B{}^C(\vec{k}^\pm) - \frac{1}{2}\gamma_{AB} R^{\mu\nu} k^\pm_\mu k^\pm_\nu - C^\mu{}_\nu{}^\rho{}_\sigma k^\pm_\mu k^\pm_\rho e^\nu_A e^\sigma_B \right)$$
$$+\xi^\pm \left(\frac{\overline{R}}{2}\gamma_{AB} + \frac{1}{2} H_\mu K^\mu_{AB} - 2K_{C(A}(\vec{k}^+) K_{B)}{}^C(\vec{k}^-) \right.$$
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- Here $s_A := k^+_\mu e^\nu_A \nabla_\nu k^{-\mu}$ is the normal connection one-form.

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- Given that the initial S is assumed to be shear-free, this property is kept at first order after deformation if and only if

$$\left[\boldsymbol{K}(\vec{k}^{+}), \delta_{\vec{\xi}}\boldsymbol{K}(\vec{k}^{-})\right] + \left[\delta_{\vec{\xi}}\boldsymbol{K}(\vec{k}^{+}), \boldsymbol{K}(\vec{k}^{-})\right] = \mathbf{0}$$

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- Taking into account that the red terms shown in the variations $\delta_{\vec{\xi}}\boldsymbol{K}(\vec{k}^{\pm})$ commute with $\boldsymbol{K}(\vec{k}^{\pm})$, this expression reduces to

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where $\delta\boldsymbol{K}^{\pm}$ is a shorthand for

$$\begin{aligned}\delta\boldsymbol{K}^{\pm} := &-\overline{\nabla}_A\overline{\nabla}_B\xi^{\pm} \pm 2s_{(A}\overline{\nabla}_{B)}\xi^{\pm} + \xi^{\mp}C^{\mu}{}_{\nu}{}^{\rho}{}_{\sigma}k^{\pm}_{\mu}k^{\pm}_{\rho}e^{\nu}_{A}e^{\sigma}_{B}\\ &+\xi^{\pm}\left(\pm\overline{\nabla}_{(A}s_{B)} - s_A s_B - \frac{1}{2}R_{\mu\nu}e^{\mu}_{A}e^{\nu}_{B}\right)\end{aligned}$$

Deforming shear-free MTS keeping zero expansion

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- Observe that one can isolate ξ^- from the last expression, and then, the other condition becomes a single equation for the remaining unknown ξ^+.

Thank you