Complexified MOTS-stability operator: an approach to the MOTS spectral problem

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1. Motivation and Problem: spectrum of the MOTS-stability operator

2. Complexifying the MOTS operator: a short-cut through the quantum particle

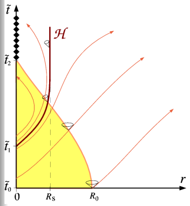
3. Conclusions and Perspectives

Scheme

Motivation: late time gravitational collapse dynamics

Establishment's picture of the gravitational collapse

1. **Singularity Theorems** (*Theorem*): trapped surfaces [Penrose 65, Hawking 67, Hawking & Penrose 70, Hawking & Ellis 73]
2. **(Weak) Cosmic Censorship** (*Conjecture*): BH horizon [Penrose 69]
3. **Spacetime driven to stationarity** (*Conjecture*)
4. **BH uniqueness** (*"Theorem"*) [e.g. Chruściel et al. 12]



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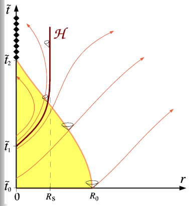
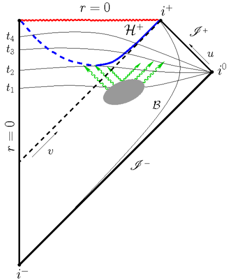


Late time evolution of Apparent Horizons

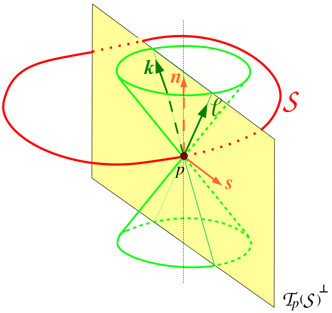
A particular probe into future timelike infinity i^+ in black hole spacetimes: **Understanding generic features of the asymptotics of Apparent Horizon worldtubes to the Event horizon.**



Marginally outer trapped surfaces (MOTS)

Let $\mathcal{S}$ be an orientable closed spacelike (codimension 2) surface with induced metric q_{ab}:

- Normal plane spanned null vectors ℓ^a and k^a
  Normalization: $\ell^a k_a = -1$
  Boost-rescaling freedom:
  $\boxed{\ell'^a = f\ell^a, k'^a = f^{-1}k^a}$, with $f > 0$
- Define the expansions:
  $\theta^{(\ell)} \equiv q^{ab}\nabla_a \ell_b = \frac{1}{\sqrt{q}}\mathcal{L}_\ell \sqrt{q}$
- **Marginally Outer Trapped Surface (MOTS)**: $\boxed{\theta^{(\ell)} = 0}$



MOTS-stability operator

Evolution of MOTS

Evolution vector h^a on $\mathcal{H}$:

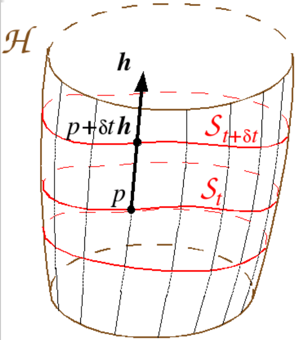
i) h^a tangent to $\mathcal{H}$ and orthogonal to $\mathcal{S}_t$.

ii) h^a transports $\mathcal{S}_t$ to $\mathcal{S}_{t+\delta t}$: $\mathcal{L}_h t = 1$.

iii) $h^a = \ell^a - C k^a$
($C > 0$: spacelike, $C = 0$: null, $C < 0$: timelike)

Then, the MOTS characterization $\delta_h \theta^{(\ell)} = 0$ is:

$$\boxed{L_{\mathcal{S}} C = \sigma^{(\ell)}_{ab} {\sigma^{(\ell)}}^{ab} + 8\pi T_{ab} \ell^a \ell^b}$$



MOTS stability operator [Andersson, Mars & Simon 05, 08]: not self-adjoint

$$\boxed{L_{\mathcal{S}} = -\Delta + 2\Omega^{(\ell)}_a D^a - \left(\Omega^{(\ell)}_a {\Omega^{(\ell)}}^a - D^a \Omega^{(\ell)}_a - \frac{1}{2} R_{\mathcal{S}} + G_{ab} k^a \ell^b \right)}$$

MOTS-spectral problem

Spectral characterization of MOTS-stability

Existence evolution of a MOTS $\mathcal{S}$: **outermost stability**.

A MOTS $\mathcal{S}$ stable iff [Andersson, Mars & Simon 05, 08]:

$$\boxed{\lambda_o > 0}$$

Principal eigenvalue λ_o: eigenvalue of $L_{\mathcal{S}}$ with smallest real part (λ_o is real).

"Can one 'hear' the dynamical response of a Black Hole horizon?" [cf. e.g. Kac 66]:

$$L_{\mathcal{S}} C \quad = \quad \sigma^{(\ell)}_{ab} {\sigma^{(\ell)}}^{ab} + 8\pi T_{ab} \ell^a \ell^b = S(\theta, \phi, t)$$

Problem proposal: *Systematic* **full spectrum analysis of** $L_{\mathcal{S}}$ [Racz 13, JLJ 15...] *for MOTS dynamics as a probe into future timelike infinity* i^+ *in BH spacetimes.*

$$\boxed{L_{\mathcal{S}} \psi_n = \lambda_n \psi_n}$$

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From MOTS-stability to the quantum charged particle

MOTS-stability operator

The operator $L_{\mathcal{S}}$ is not self-adjoint:

$$L_{\mathcal{S}} = -\Delta + 2{\Omega^{(\ell)}}^a D_a - \left(\Omega^{(\ell)}_a {\Omega^{(\ell)}}^a - D^a \Omega^{(\ell)}_a - \frac{1}{2} R_{\mathcal{S}} + G_{ab} k^a \ell^b \right)$$

Structural similarity with the quantum charged particle (QCP) [JLJ 15]

$$\boxed{\Omega^{(\ell)}_a \to \frac{iq}{\hbar c} A_a \quad , \quad R_{\mathcal{S}} \to \frac{4mq}{\hbar^2} \phi \quad , \quad G_{ab} k^a \ell^b \to -\frac{2m}{\hbar^2} V}$$

the MOTS-stability operator becomes $\frac{\hbar^2}{2m} L_{\mathcal{S}} \to \hat{H}$ where

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + \frac{i\hbar q}{mc} A^a D_a + \frac{i\hbar q}{2mc} D_a A^a + \frac{q^2}{2mc^2} A_a A^a + q\phi + V$$

From MOTS-stability to the quantum charged particle

MOTS-stability operator

The operator $L_{\mathcal{S}}$ is not self-adjoint:

$$\left[-\left(D-\Omega^{(\ell)}\right)^2+\frac{1}{2}R_{\mathcal{S}}-G_{ab}k^a\ell^b\right]\psi=\lambda\psi$$

Structural similarity with the quantum charged particle (QCP) [JLJ 15]

$$\hat{H}=\frac{1}{2m}\left(-i\hbar D-\frac{q}{c}A\right)^2+q\phi+V$$

Hamiltonian operator of a **non-relativistic (spin-0) quantum particle of mass m and charge q** moving on $\mathcal{S}$ under a magnetic and electric fields with vector and scalar potentials given by A^a and ϕ, and an additional external potential V.

MOTS $\mathbb{R}^+$-gauge transformations

Lemma [JLJ 15]. *Under the null normal ($\mathbb{R}^+$-gauge) rescaling $\ell'^a = f\ell^a$, $k'^a = f^{-1}k^a$, with $f > 0$:*

i) *The expansion and Hájíček form Ω_a transform as*

$$\theta^{(\ell')} = f\theta^{(\ell)} \quad , \quad \Omega'_a = \Omega_a + D_a(\ln f) \ .$$

ii) *The MOTS-stability operator transforms covariantly*

$$(L_{\mathcal{S}})'\psi = fL_{\mathcal{S}}(f^{-1}\psi) \ ,$$

where $(L_{\mathcal{S}})'\psi \equiv \delta_{\psi(-k')}\theta^{(\ell')}$.

iii) *The MOTS-eigenvalue problem is invariant under the additional eigenfunction transformation, $\psi' = f\psi$*

$$L_{\mathcal{S}}\psi = \lambda\psi \quad \rightarrow \quad (L_{\mathcal{S}})'\psi' = \lambda\psi' \ .$$

*(***Compare with the $U(1)$-gauge invariance of the Schrödinger equation***)*

MOTS-stability and quantum charged particle similarities

Spectral problem: $L_{\mathcal{S}} \leftrightarrow \hat{H}$

$L_{\mathcal{S}}\psi = \lambda\psi$ (MOTS) , $\hat{H}\psi = E\psi$ (stationary quantum charged particle)

MOTS stability and quantum stability

$\lambda_o \geq 0$ and E bounded below

Abelian gauge symmetry

$$
\begin{array}{lllll}
A_a & \rightarrow & A_a - D_a\sigma & , \quad \psi \rightarrow e^{iq\sigma/(c\hbar)}\psi & , \quad \text{(quantum charged particle)} \\
\Omega_a^{(\ell)} & \rightarrow & \Omega_a^{(\ell)} - D_a\sigma & , \quad \psi \rightarrow f\psi = e^{-\sigma}\psi & , \quad \text{(MOTS-spectral problem)}
\end{array}
$$

Phase $U(1)$ (charged particle) and rescaling $\mathbb{R}^+$ (MOTS) gauge symmetries.

Operators obtained by "minimal coupling" of the gauge potentials

$$
i\hbar\partial_t \rightarrow i\hbar\partial_t - q\phi \quad , \quad -i\hbar D_a \rightarrow -i\hbar D_a - \frac{q}{c}A_a
$$
$$
D_a \rightarrow D_a - \Omega_a^{(\ell)}
$$

MOTS-stability operator and the fine-structure constant α

Analyticity Conjecture

Given an orientable closed surface $\mathcal{S}$ and the one-parameter family of operators

$$
\begin{aligned}
L_{\mathcal{S}}[\sqrt{\alpha}] &= -\left(D - i\sqrt{\alpha}A\right)^2 - \alpha\tilde{\phi} + V \\
&= -\Delta + 2i\sqrt{\alpha}A^a D_a + i\sqrt{\alpha}D^a A_a + \alpha A_a A^a - \alpha\tilde{\phi} + V
\end{aligned}
$$

in the (squared-root) of the fine-structure constant $\alpha \equiv \frac{e^2}{\hbar c}$:
the MOTS-spectrum ($\alpha = -1$) can be recovered as an "analytic continuation" of the spectrum in the quantum charged particle spectrum ($\alpha = 1$) self-adjoint problem.

Stable MOTS as QCP of negative fine-structure constant α

If the Analyticity Conjecture proves valid...

The MOTS-stability spectrum problem is "essentially" reduced to that of the self-adjoint problem of the stationary non-relativistic quantum charged particle.

Test 1: principal eigenvalue/ground state characterization

Ground state energy of the quantum charge particle from MOTS expression

A **gauge-invariant** characterization of the ground state E_o would be given by [Andersson, Mars & Simon 08]

$$\boxed{E_o = \inf_{\psi>0} \int_{\mathcal{S}} \left(|D\psi|^2 + \left(e\phi + V + |D\omega_\psi + z|^2\right)\psi^2\right) dA}$$

where $A_a = z_a + D_a f$ (with $D_a z^a = 0$, for any closed Riemannian $\mathcal{S}$), $\int_{\mathcal{S}} \psi^2 dA = 1$ and ω_ψ satisfies, for a given $\psi > 0$

$$-\Delta\omega_\psi - \frac{2}{\psi} D_a\psi D^a\omega_\psi = \frac{2}{\psi} z^a D_a\psi$$

Then:

- The expression for E_o **can be proved to actually hold** [JLJ 15]: from standard Rayleigh-Ritz formula for the selfadjoint Hamiltonian $\hat{H}$.
- Expression gauge invariant and paramagnetic term recast as a diamagnetic one.

Test 2: explicit example ("Landau levels")

MOTS case

With $q_{ab} = d\theta^2 + \sin^2\theta d\varphi^2$ and $\Omega^{(\ell)}_a = (0, a\sin^2\theta)$:

$$L^a_{\mathcal{S}} = \left[-\Delta + 2a{\Omega^{(\ell)}}^a D_a + aD^a\Omega^{(\ell)}_a - a^2\Omega^{(\ell)}_a{\Omega^{(\ell)}}^a + \frac{1}{2}R_{\mathcal{S}} - G_{ab}k^a\ell^b\right]$$

MOTS: $\boxed{\lambda = (\lambda_{\ell m} + 1 - a^2) + i2am \quad , \quad \psi = S_{lm}(a, \cos\theta)e^{im\varphi} \quad \textbf{(prolate)}}$

QCP case: complex rotation $a \to ia$

With $q_{ab} = d\theta^2 + \sin^2\theta d\varphi^2$ and $\Omega^{(\ell)}_a = (0, ia\sin^2\theta)$:

$$L^{ia}_{\mathcal{S}} = \left[-\Delta + 2ia{\Omega^{(\ell)}}^a D_a + iaD^a\Omega^{(\ell)}_a + a^2\Omega^{(\ell)}_a{\Omega^{(\ell)}}^a + \frac{1}{2}R_{\mathcal{S}} - G_{ab}k^a\ell^b\right]\psi$$

QCP: $\boxed{\lambda = (\lambda_{\ell m} + 1 + a^2) - 2am \quad , \quad \psi = S_{lm}(ia, \cos\theta)e^{im\varphi} \quad \textbf{(oblate)}}$

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Conclusions and perspectives

Conclusions

- **Analyticity conjecture** on $L_{\mathcal{S}}$: reduction of MOTS-spectral problem to a self-adjoint problem (quantum charged particle).
- Non-trivial tests of the conjecture.
- Study of **Kerr-Newman case** [Racz et al. 13]: perturbation approach (slowly rotating and extremal case).

Perspectives

- Towards a **spinorial approach** to MOTS stability and MOTS-spectrum.
- Introduction of **semi-classical tools**:

$$H_{\rm cl}[\sqrt{\alpha}] = (p - \sqrt{\alpha}\Omega^{(\ell)})^2 + \frac{1}{2}R_{\mathcal{S}} - G_{ab}k^a\ell^b$$

- **Late time apparent horizon dynamics** (*"poor's man"*) aim: identification of spectral properties of $L_{\mathcal{S}}$ which are generic for apparent horizons with "good" asymptotic behaviour at i^+.