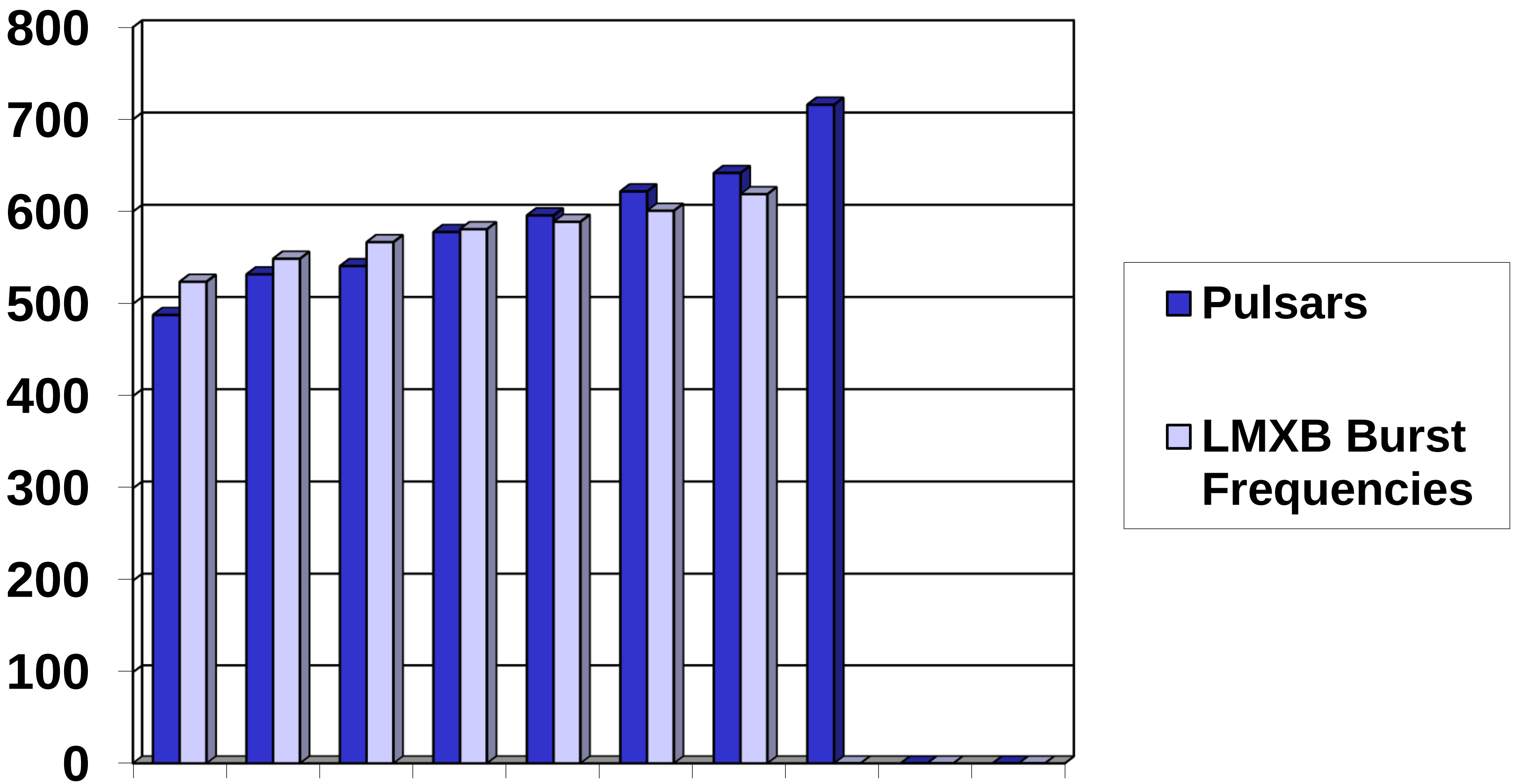
Can magnetic-field windup kill the r-mode instability of neutron stars?

John Friedman, Lee Lindblom,
Luciano Rezzolla, Andrey Chugunov

Parker Center for Gravitation, Cosmology, and Astrophysics

All or nearly all of the fastest spinning neutron stars we observe are not young stars but are instead old neutron stars that have been spun up by accretion from a companion.

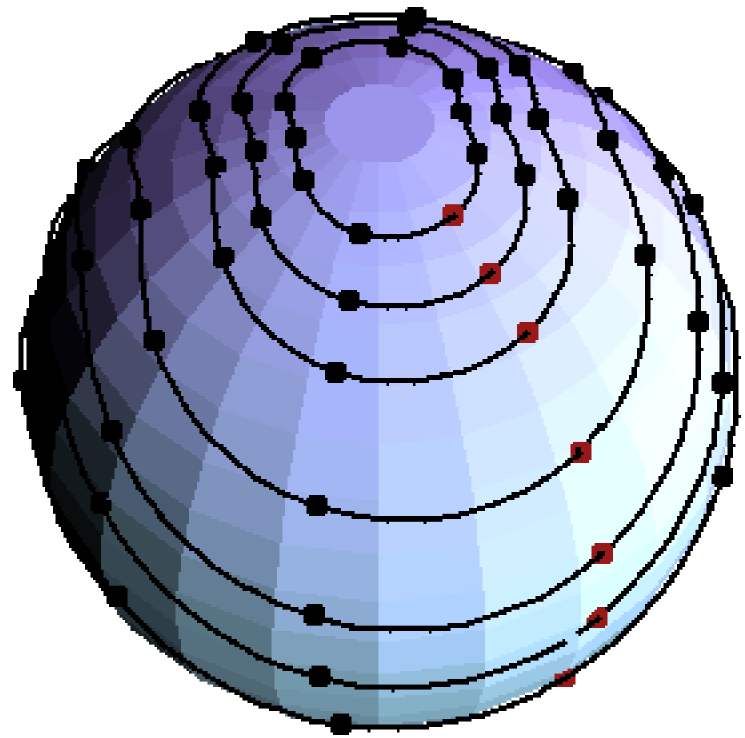
The angular velocities of observed neutron stars show a cutoff below 800 Hz.



This limit on spin may be set by a gravitational-wave driven (CFS) instability of an *r*-mode – a perturbation of the fluid velocity

$$\delta \mathrm{v} \propto \mathbf{r} \times \nabla Y_{22}\ e^{\beta t}$$

For old accreting stars, the growth time β^{-1} is months or years.

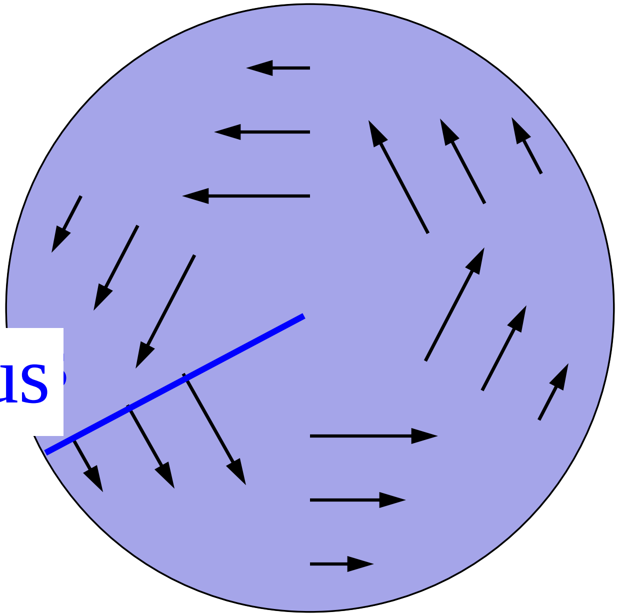


Animation by Chad Hanna

At second-order in perturbation theory, radiation-reaction and quadratic terms in the perturbed Euler equation drive an exponentially growing differential rotation.

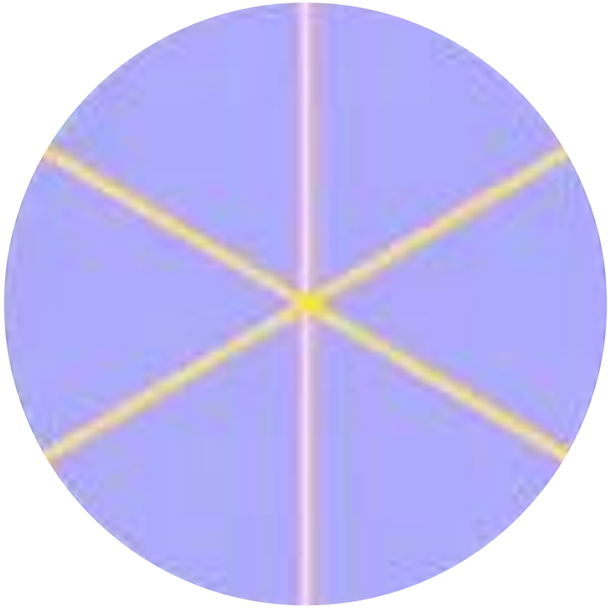
$$\delta\phi = \varphi(\varpi)\, e^{2\beta t}$$

$$\varpi = \text{cylindrical radius}$$



Friedman, Lindblom, Lockitch'16, following Sa '04 (stable mode), Levin & Ushormirsky '00 (toy model)

The differential rotation winds up a background magnetic field



$$\delta B \sim B_0 \delta\phi$$

Spruit '99
Rezzola, Lamb, Shapiro '00, '01, & Markovic '01
Cuofano, Drago '10, Cuofano, Dall'Osso, Drago, Stella '12,
Chugunov '15

Stuart Shapiro, James Cook, Branson Stephens

And the growing magnetic field damps the r-mode instability when

$$\frac{dE_{linear\,\mathrm{mod}\,e}}{dt} = \frac{dE_{magnetic}}{dt}$$

$$\delta B \sim 10^{12}\,G \qquad \alpha_{-4} R_6 \Omega_3 \sqrt{\rho_{14.5}}$$

Previous studies find damping of the instability in newborn stars with an initial $B_0 > 10^{10}$ G and damping or significant alteration of the unstable mode for old accreting neutron stars.

Revisit: Two changes

1. Early studies of newborn neutron stars looked at proto-neutron stars with large magnetic fields and assumed a large saturation amplitude (amplitude at which coupling to other modes stops the growth of the r-mode)

$$\frac{1}{100} < \alpha_{\mathrm{saturation}} < 1, \qquad \text{with amplitude defined by } \delta \mathrm{v} = \alpha\,\Omega$$

But a subsequent series of papers in 2nd-order perturbation theory finds

$$\alpha_{saturation} < 10^{-4}$$

(Arras, Bondarescu, Brink, Morsink, Teukolsky, Wasserman)

2. Include back-reaction of growing magnetic field on growth of differential rotation.

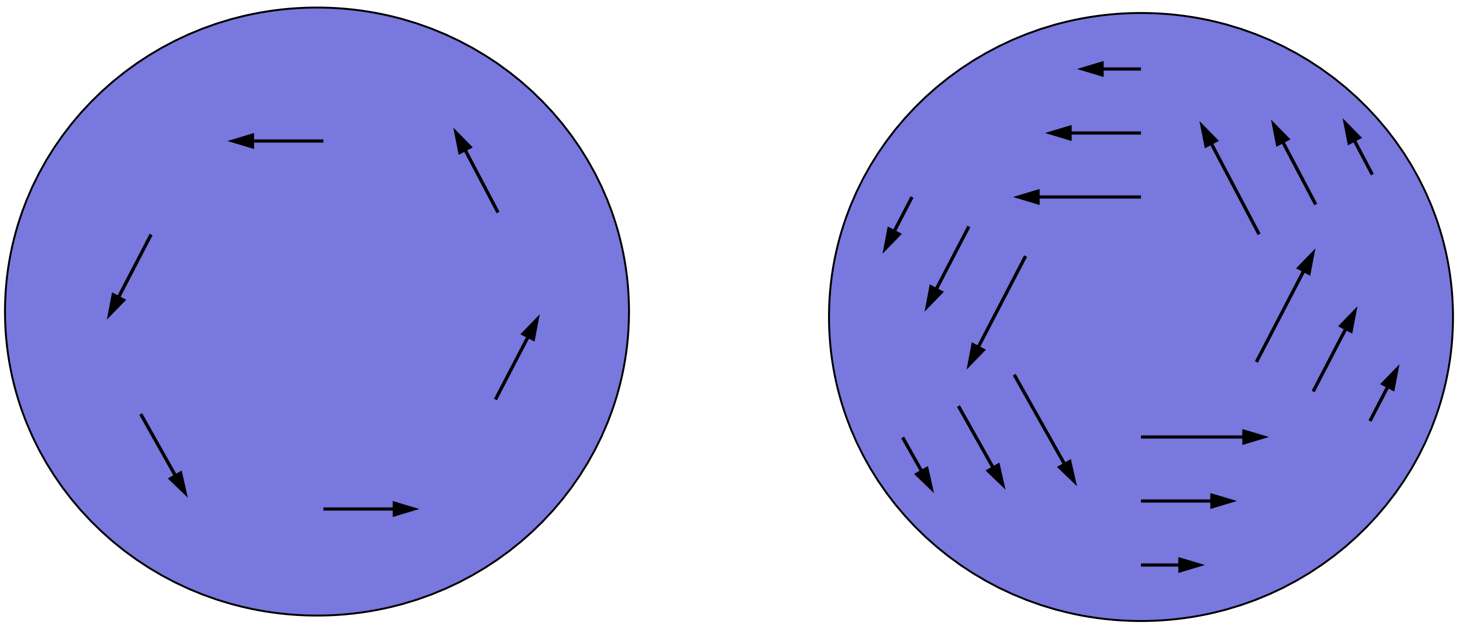
A 10^{12} G field is too small to change the shape or frequency of the linear r-mode.
(Morsink, Rezania, ’02; Lee ’ 05, Glampedakis, Andersson ’ 07,S. Abbassi, M. Rieutord; Lander, Jones, Passamonti, Lander, D. I. Jones, and A. Passamonti, ’10, V. Rezania ’12, Chirenti and J. Sk´akala ’13)

But including even an initial 10^8 G has a dramatic effect on the maximum growth of differential rotation.

To see this, first look at adding differential rotation to a rotating star.

$$\Omega \quad \rightarrow \quad \Omega + \delta\Omega(\varpi),$$

$$\varpi = \text{cylindrical radius}$$



Because the perturbed star is a nearby equilibrium, the perturbation is time-independent.

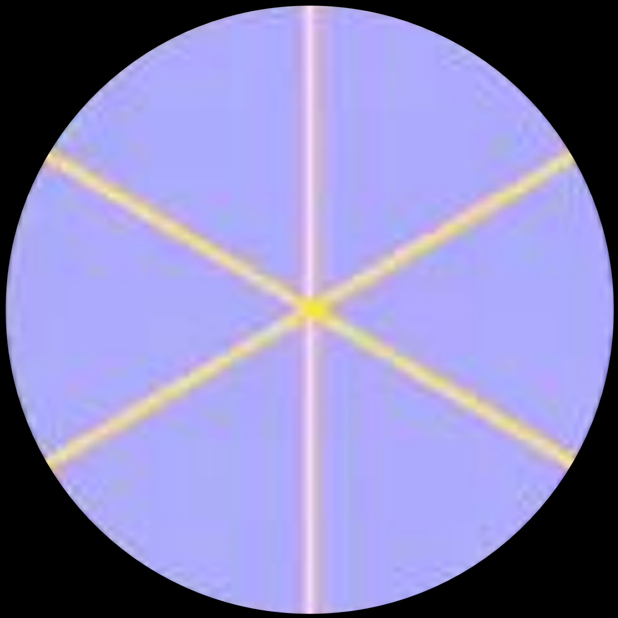
But for an equilibrium star with a magnetic field, adding initial differential rotation is a periodic perturbation:

Field lines wind and unwind with timescale the Alfvén time, time for wave in magnetic field to cross neutron star

$$t_A \sim 10^6\,\mathrm{s}\ \sqrt{10^9 G / B}$$

$$\omega_A \sim 10^{-5}\,\mathrm{s}^{-1}$$

Here's what happens in a toy model invented by Stuart Shapiro, the star represented by differentially rotating fluid in a cylinder with an initially radial magnetic field. The solution is exact.



Stuart Shapiro, James Cook, Branson Stephens
http://research.physics.illinois.edu/cta/movies/MBRAKING/INCOMPR/evolution.html

Driven system: a dramatic difference between a driven mode with zero-frequency and a driven mode with nonzero frequency when $\omega_A > \beta$:

Before nonlinear saturation

zero-frequency

$$\partial_t \delta\Omega = \partial_t^{\,2} \delta\phi \;=\; f_0\, e^{2\beta t}$$

$$\delta\phi = \frac{f_0}{4\beta^2} e^{2\beta t}$$

frequency ω_A

$$\partial_t^{\,2} \delta\phi + \omega_A^{\,2} \delta\phi \;=\; f_0\, e^{2\beta t}$$

$$\delta\phi \sim \frac{f_0}{\omega_A^{\,2}} e^{2\beta t} \quad \text{for } \omega_A >> \beta$$

Angular displacement reduced to

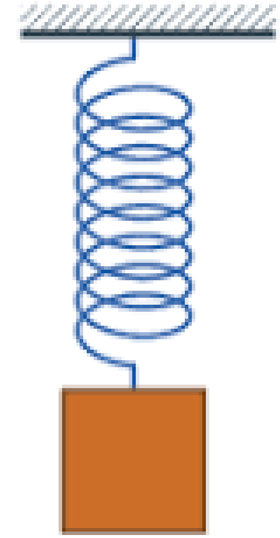
$$\delta\phi_{\text{saturation}} \sim 10^{-1} \sqrt{\frac{B}{10^9\, G}}$$

After nonlinear saturation

zero-frequency

A constant force gives constant acceleration

frequency ω_A



$\delta\phi$ oscillates about saturation value

Angular displacement now limited for a normal core to

$$\delta\phi < 2\ \delta\phi_{\text{saturation}}$$

This estimate may greatly overstate $\delta\phi_{max}$ because the Alfvén frequency of a type II superconductor is much higher: NS core is probably superconducting, with flux tubes carrying field $H_c \sim 10^{16}$ G

$$\omega_{\mathrm{A\ normal}} = \frac{B}{R\sqrt{4\pi\rho}} \sim 10^{-6}\ s^{-1} \rightarrow$$

$$\omega_{\mathrm{A superconductor}} = \frac{\sqrt{BH_c}}{R\sqrt{4\pi\rho}} \sim 10^{-4}\ s^{-1} \frac{B_{10}}{R_6\sqrt{\rho_{15}}}$$

$$\boxed{\delta\phi_{\mathrm{max,superconductor}} < 10^{-5}\ B_{10}^{-1} \alpha_{-4}^{\ 2}\ !}$$

Conclusion

Observed magnetic fields in X-ray binaries and double neutron star systems older than 10^7 yr have

$B<10^{11}$ G.

The differential rotation possible at that field size is too small to significantly alter the linear r-mode or to drive the field to 10^{12} G.

Caveats:

Calculations done for a discrete spectrum and assuming no zero-frequency modes that wind up the magnetic field, and neglecting MRI instability.

Details:

Equation governing the 2nd order r-mode is

$$\ddot{\xi}^a + B^a{}_b \dot{\xi} + C^a{}_b \xi^b$$

$$= f^a{}_{GR} + \text{terms quadratic in first-order perturbation}$$

B is anti-self adjoint, C self-adjoint and they do not commute.

With discrete modes, nevertheless have a spectral decomposition of the form

$$\xi^a = \sum_n \frac{\left\langle \hat{\xi}_n, f + Q \right\rangle}{2\beta - i\omega_n} \hat{\xi}_n + c.c.$$

with the mode functions $\hat{\xi}_n$ normalized by a conserved symplectic product.

The contribution from the nth mode is again of order

$$\frac{1}{\kappa_A(4\beta^2+\omega_A^2)}(f+Q)$$

with

$$\kappa_A = 1 - 2\frac{\Omega}{\omega_A}\Im\frac{\int dV\rho\xi^{\varpi *}\xi^{\hat{\phi}}}{\int dV\rho|\xi|^2}.$$

MHD Euler equation

$$\partial_t \mathrm{v} + \mathrm{v}\cdot\nabla\mathrm{v} + \frac{\nabla p}{\rho} + \nabla\Phi - \frac{1}{4\pi\rho}(\nabla\times B)\times B = f_{GR}$$

Nonlinear perturbation:

$$\partial_t \delta\mathrm{v} + \delta\mathrm{v}\cdot\nabla\mathrm{v} + \mathrm{v}\cdot\nabla\,\delta\mathrm{v} + \delta\mathrm{v}\cdot\nabla\,\delta\mathrm{v} + \cdots = \delta f_{GR}$$

The linear perturbation (r-mode) is an m=2 mode, proportional to $\cos(2\phi-\omega t)e^{\beta t}$

$$\partial_t \delta^{(2)}\mathrm{v} + \delta^{(2)}\mathrm{v}\cdot\nabla\mathrm{v} + \mathrm{v}\cdot\nabla\,\delta^{(2)}\mathrm{v} + \cdots = \delta^{(2)} f_{GR} - \delta\mathrm{v}\cdot\nabla\,\delta\mathrm{v} + \cdots$$

$$= \delta^{(2)} f_{GR} + \text{terms quadratic in first-order perturbation}$$

MHD Euler equation

$$\partial_t \mathrm{v} + \mathrm{v}\cdot\nabla\mathrm{v} + \frac{\nabla p}{\rho} + \nabla\Phi - \frac{1}{4\pi\rho}(\nabla\times B)\times B = f_{GR}$$

Nonlinear perturbation:

$$\partial_t \delta\mathrm{v} + \delta\mathrm{v}\cdot\nabla\mathrm{v} + \mathrm{v}\cdot\nabla\,\delta\mathrm{v} + \delta\mathrm{v}\cdot\nabla\,\delta\mathrm{v} + \cdots = \delta f_{GR}$$

The linear perturbation (r-mode) is an m=2 mode, proportional to

$$\cos(2\phi - \omega t)e^{\beta t}$$

The nonlinear perturbation has an axisymmetric part:

$$\partial_t \delta \mathrm{v} + \delta \mathrm{v}\cdot\nabla \mathrm{v} + \mathrm{v}\cdot\nabla \delta \mathrm{v} + \cdots = \delta f_{GR} - \delta \mathrm{v}\cdot\nabla \delta \mathrm{v} + \cdots$$

$$= \delta f_{GR} + \text{terms quadratic in the perturbation}$$

and terms quadratic in the first-order perturbation are

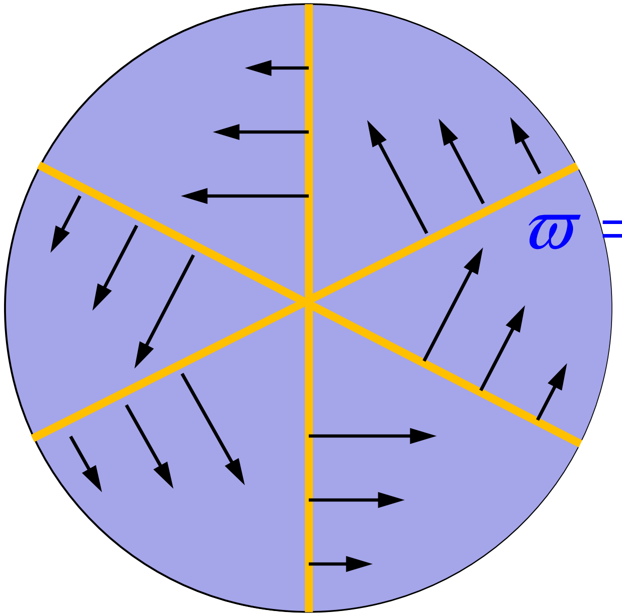
a sum of m=0 and m=4 parts

$$[\cos(2\phi - \omega t) e^{\beta t}]^2 = \frac{1}{2} e^{2\beta t} + \frac{1}{2}\cos(4\phi - 2\omega t)\, e^{2\beta t}$$

The axisymmetric part of the nonlinear r-mode is exponentially growing differential rotation:

$$\partial_t \delta \mathrm{v}^{\phi} \equiv \partial_t \delta \Omega \; = \; f \, e^{2\beta t}$$

$$\delta \Omega \propto e^{2\beta t}$$



ϖ = cylindrical radius