The Conformal Constraint Equations as an Evolution System

Jarrod Williams

(Joint work with J.A.Valiente Kroon)

Queen Mary University of London

j.l.williams@qmul.ac.uk

July 12, 2016

Introduction

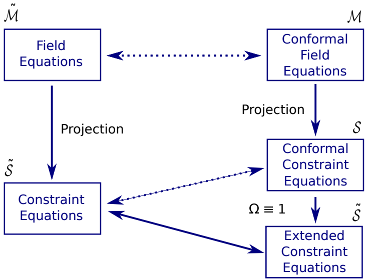
- Solutions to the constraint equations describe **initial data** for the Einstein field equations.
  i) How in practice are the constraints solved?
  ii) Which fields comprise the *freely-prescribable* data for the system?
- The constraints are usually recast as an elliptic system, but is this the only approach to answering the above questions?

The Constraint Equations

- The Hamiltonian (**H**) and Momentum (**M**) constraints are given by

$$\textbf{(H):} \quad \tilde{r}[\tilde{h}] + \tilde{k}^2 - \tilde{K}_{ij}\tilde{K}^{ij} = 0,$$
$$\textbf{(M):} \quad \tilde{D}^i\tilde{K}_{ij} - \tilde{D}_j\tilde{k} = 0$$

where $\tilde{k} \equiv \tilde{K}_i{}^i$. $\tilde{h}_{ij}$ a (neg. def.) Riemannian 3-metric and $\tilde{K}_{ij}$ the *extrinsic curvature*.



- Highly undetermined (4 equations for 12 unknowns)
- Methods of Solution:
  - The Conformal Method applied to the constraint equations
  - The Butscher Method, applied to the extended constraint equations
  - As an evolution system

I. Racz, Classical Quant Grav **33**, *1 (2016), 015014*

The Conformal Field Equations (CFEs)

- Under a conformal change $g = \Xi^2 \tilde{g}$, vacuum EFEs become:

$$R[\tilde{g}]_{ab} = \Xi^{-2}(3\nabla^c\Xi\nabla_c\Xi - \Xi\Box\Xi)g_{ab} - \Xi^{-1}\nabla_a\nabla_b\Xi$$

Singular at conformal boundary $\mathscr{I}$ ($\Xi = 0$).

The CFEs - H.Friedrich, Comm. Math. Phys. 91 (1983).

$$\nabla_a\nabla_b\Xi = -\Xi L_{ab} + s g_{ab} \tag{1a}$$

$$\nabla_a s = -L_{ab}\nabla^b\Xi \tag{1b}$$

$$\nabla_c L_{db} - \nabla_d L_{cb} = d_{abcd}\nabla^a\Xi \tag{1c}$$

$$\nabla^a d_{abcd} = 0 \tag{1d}$$

$$6\Xi s - 3\nabla_c\Xi\nabla^c\Xi = 0 \tag{1e}$$

where $\quad L_{ab} := \frac{1}{2}R_{ab} - \frac{1}{12}R g_{ab}, \quad d_{abcd} := \Xi^{-1}C_{abcd}, \quad s := \frac{1}{4}\Box\Xi + \frac{1}{24}R\Xi$

- *Regular* at $\mathscr{I}$ and Ξ now an independent variable
- The *algebraic constraint* (1e) need only be verified at a single point.

The Conformal Constraint Equations

Given a spinor dyad $\epsilon_{\boldsymbol{A}}{}^{A}$, and assuming:

i) *umbilical* physical data $\tilde{K}_{\boldsymbol{ABCD}} = \frac{1}{3}\tilde{k}\tilde{h}_{\boldsymbol{ABCD}}$ ($\Rightarrow \tilde{k}$ const.),

ii) a *maximal* conformal representative, $k = 0$,

the CCEs are equivalent to the vanishing of the following *zero quantities*

$$\begin{aligned}
Q_{\boldsymbol{AB}} &:= D_{\boldsymbol{AB}}\Omega - \Sigma_{\boldsymbol{AB}} && \text{(2a)}\\
Z_{\boldsymbol{ABCD}} &:= D_{\boldsymbol{AB}}\Sigma_{\boldsymbol{CD}} + \Omega L_{\boldsymbol{ABCD}} - s h_{\boldsymbol{ABCD}} && \text{(2b)}\\
Z_{\boldsymbol{AB}} &:= D_{\boldsymbol{AB}}s + L_{\boldsymbol{ABCD}}\Sigma^{\boldsymbol{CD}} && \text{(2c)}\\
\Delta_{\boldsymbol{ABCD}} &:= D^{\boldsymbol{P}}{}_{(\boldsymbol{A}}L_{\boldsymbol{B})\boldsymbol{PCD}} - d_{\boldsymbol{PAB}(\boldsymbol{C}}\Sigma_{\boldsymbol{D})}{}^{\boldsymbol{P}} && \text{(2d)}\\
\Lambda_{\boldsymbol{AB}} &:= D^{\boldsymbol{AB}}d_{\boldsymbol{ABCD}} && \text{(2e)}\\
\Sigma_{\boldsymbol{AB}}{}^{\mathfrak{a}} &:= D^{\boldsymbol{P}}{}_{(\boldsymbol{A}}e_{\boldsymbol{B})\boldsymbol{P}}{}^{\mathfrak{a}} - \frac{1}{2}C_{\mathfrak{b}}{}^{\mathfrak{a}}{}_{\mathfrak{c}}\, e_{\boldsymbol{PA}}{}^{\mathfrak{b}} e_{\boldsymbol{B}}{}^{\boldsymbol{P}\mathfrak{c}} && \text{(2f)}\\
\Xi_{\boldsymbol{ABCD}} &:= D^{\boldsymbol{P}}{}_{(\boldsymbol{A}}\gamma_{\boldsymbol{B})\boldsymbol{PCD}} + \gamma_{\boldsymbol{DQP}(\boldsymbol{A}}\gamma_{|\boldsymbol{C}|}{}^{\boldsymbol{QP}}{}_{\boldsymbol{B})} \\
&\quad + \frac{1}{2}\left(L_{(\boldsymbol{ABCD})} + \Omega d_{\boldsymbol{ABCD}}\right) - \frac{1}{3}L h_{\boldsymbol{ABCD}} && \text{(2g)}
\end{aligned}$$

to be solved for $(\Omega, \Sigma, s, L, d, e^{\mathfrak{a}}, \gamma)$.

Decomposition of the CCEs

- Let ρ_{AB} be s.t. $\rho^{AB}\rho_{AB} = -2$, defining an orthogonal *distribution*.
- Assuming ρ_{AB} is *twistfree*, $\chi^{P}{}_{(AB)P} = 0$, then $\mathcal{S}$ is foliated by leaves $\$_r$ with induced metric

$$\zeta_{ABCD} := \tfrac{1}{2}\rho_{AB}\rho_{CD} + h_{ABCD}$$

- Any spinor decomposes into orthogonal and transverse parts:

$$\kappa_{AB} \equiv -\tfrac{1}{2}\rho_{AB}\kappa^{\perp} + \rho_{A}{}^{P}\kappa^{\parallel}_{BP}, \quad \kappa^{\perp} := \rho^{PQ}\kappa_{PQ}, \quad \kappa^{\parallel}{}_{AB} := \rho_{(A}{}^{Q}\kappa_{B)Q}$$

- Defining $\mathcal{P} := \rho^{AB}D_{AB}$ and the *Sen* derivative $\mathcal{D}_{AB} := \rho_{A}{}^{P}D_{BP}$,

$$D_{AB} \equiv -\tfrac{1}{2}\rho_{AB}\mathcal{P} + \rho_{A}{}^{P}\mathcal{D}_{BP}$$

Extrinsic Curvature: $\chi_{ABCD} \equiv \frac{1}{\sqrt{2}}\rho_{D}{}^{P}\mathcal{D}_{AB}\rho_{CP}$

Acceleration: $\chi_{AB} \equiv \frac{1}{\sqrt{2}}\rho_{B}{}^{P}\mathcal{P}\rho_{AP}$

Decomposition of the CCEs - Leaf Constraints

- Decomposing the CCEs with respect to ρ_{AB} gives **propagation equations** (the orthogonal parts) and **leaf constraints** (the transverse parts).

$$\mathcal{D}_{AB}\Omega + \text{l.o.t.} = 0$$

$$\mathcal{D}^{P}{}_{(A}e_{B)P}{}^{\mathfrak{a}} + \text{l.o.t.} = 0$$

- Equation (2e) for d_{ABCD} has no constraint part
- To satisfy the algebraic condition (1e), it is sufficient to impose at a single point on a given leaf, $p \in \mathbb{S}_r$, the following

Algebraic Constraint: $$\Sigma_{AB}\Sigma^{AB} - 2\Omega s = -\tfrac{1}{9}\tilde{k}^2 \qquad (3)$$

- If $\tilde{\mathbb{S}}_r$ is to be (outer-)trapped ($\theta^+ \leq 0$), so that data corresponds to a BH spacetime, we must impose the additional

Trapping Condition: $$\left(\Omega \mathrm{tr}_{\zeta}\chi - \sqrt{2}\,\Sigma^{\perp}\right)\Big|_{\mathbb{S}_r} \leq -\tfrac{2}{3}\tilde{k} \qquad (4)$$

Decomposition of the CCEs - Propagation Equations

- The propagation equations consist of subsystems of one of the three following forms:

$$\mathcal{P}\Omega + \text{l.o.t.} = 0 \tag{5}$$

$$\mathcal{P}\varpi^{\mathfrak{a}} + 2\mathcal{D}^{\boldsymbol{PQ}}\varpi_{\boldsymbol{PQ}}{}^{\mathfrak{a}} + \text{l.o.t.} = 0, \tag{6a}$$

$$\mathcal{P}\varpi_{\boldsymbol{AB}}{}^{\mathfrak{a}} - \mathcal{D}_{\boldsymbol{AB}}\varpi^{\mathfrak{a}} + \text{l.o.t.} = 0 \tag{6b}$$

$$\mathcal{P}\Theta_{\boldsymbol{AB}} + 2\mathcal{D}^{\boldsymbol{PQ}}\Theta_{\boldsymbol{PQAB}} + \text{l.o.t.} = 0 \tag{7a}$$

$$\mathcal{P}\Theta_{\boldsymbol{ABCD}} - \mathcal{D}_{\boldsymbol{AB}}\Theta_{\boldsymbol{CD}} + \text{l.o.t.} = 0 \tag{7b}$$

- No propagation equation for the trans.-trans. component, $d^{\|}$, of d.

Evolution Method: Given leaf constraint data on $\$_0$, evolve according to the propagation equations, with freely-prescribed $d^{\|}$ interpreted as *sources*.

i) Are the leaf constraints **propagated**? i.e. are they automatically satisfied on each $\$_r$, $r \geq r_0$?
ii) Is the problem **well-posed**? i.e. Cauchy stable?

Propagation of the leaf constraints

- CCEs equivalent to (Leaf Constraints + Propagation Equations)
- In order for candidate initial data to solve the CCEs, we require the leaf constraints to be **propagated**.

Remarkable property: Assuming the propagation equations are satisfied, the zero quantities $(Q, Z, \Delta, \Lambda, \Sigma, \Xi)$ satisfy homogeneous *subsidiary* propagation equations.

- The subsidiary equations follow from the Cartan formula:

$$\mathcal{L}_\rho \kappa = i_\rho(d\kappa) + d(i_\rho \kappa)$$

$i_\rho \kappa$ – the propagation equations

$d\kappa$ – expressible as a homogeneous combination of zero quantities

Propagation of the first structure equation

$$\mathcal{L}_\rho \Sigma_k{}^l{}_j = \rho^i (\Xi^l{}_{[kij]} - \Sigma_{[i}{}^m{}_j \Sigma_{k]}{}^l{}_m)$$

Application: Initial data for perturbations of Schwarzschild

- Umbilical Initial Data for Schwarzschild ($2m = 1$):

$$\tilde{h} = -\left(1 - \tfrac{1}{r} + \tfrac{1}{9}\tilde{k}^2 r^2\right)^{-1} dr^2 - r^2 d\sigma^2, \qquad \tilde{K} = \tfrac{1}{3}\tilde{k}\ \tilde{h}$$

- Foliated by metric spheres, $\tilde{\mathbb{S}}_r$.
- Corresponding (umbilical) Conformal Initial Data:

$$\begin{aligned}
\mathring{\Omega} &= r^{-1},\\
\mathring{e}_3 &= r\left(1 - \tfrac{1}{r} + \tfrac{1}{9}\tilde{K}^2 r^2\right)^{1/2} \frac{\partial}{\partial r},\\
\mathring{s} &= \tfrac{1}{2} r^{-1}(r^{-1} - 1),\\
\mathring{L} &= \tfrac{1}{2}(2r^{-1} - 1)(\mathring{e}_3 \otimes \mathring{e}_3) + \tfrac{1}{2}(1 - r^{-1})(\mathring{e}_1 \otimes \mathring{e}_1 + \mathring{e}_2 \otimes \mathring{e}_2),\\
\mathring{d} &= -\mathring{e}_3 \otimes \mathring{e}_3 + \tfrac{1}{2}(\mathring{e}_1 \otimes \mathring{e}_1 + \mathring{e}_2 \otimes \mathring{e}_2)
\end{aligned}$$

where $\{\mathring{e}_1, \mathring{e}_2\}$ is an orthonormal frame on $\mathbb{S}_r$.
- The data is **conformally-extendable** beyond $\mathscr{I}$.

Further Work:

- Explore Cauchy stability of the propagation system
- What about the more general (non-umbilical) case? e.g. construction of initial data for perturbations of the Kerr spacetime.

Thanks for listening!