Thermodynamic Metrics for Black Hole Physics

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Introduction

We give a brief survey of thermodynamic metrics, in particular the Ruppeiner and Weinhold geometries constructed from the Hessian of the entropy function, and how they apply to black hole thermodynamics.

We then provide a detailed discussion of the Gibbs surface of Kerr black holes. In particular, we analyze its global properties and extend it to take the entropy function of the inner horizon into account.

We find a coordinate system where the Weinhold geometry for a Kerr black hole is manifestly flat. In this coordinate system we find the geodesics of the Ruppeiner geometry. Comparison between mass and entropy illustrates the Penrose process.

If the state of a thermodynamic system is changed the least entropy increase is obtained if the change follows a geodesic in Ruppeiner geometry.

Ruppeiner and Weinhold Geometries

The Ruppeiner metric is defined on the Gibbs surface of a thermodynamic system using nothing but the fundamental relation $S = S(X)$, where S is the entropy and X^i are the remaining extensive variables of the system, including its energy. The metric tensor is simply the negative of the Hessian matrix,

$$g_{ij} \equiv -\frac{\partial^2 S}{\partial X^i \partial X^j} \equiv -S_{,ij} \ .$$

The Weinhold metric is the Hessian of the energy U as a function of the extensive variables, including the entropy.
These two metrics are conformally related,

$$ds^2 = \frac{1}{T} ds_W^2 \ .$$

where $T = \frac{\partial U}{\partial S}$ is the temperature. For black holes U is the mass M.

Kerr Black Holes, also inner horizon

We need to know that the event horizon of a Kerr black hole with mass M and angular momentum J has area:

$$A_+ = 8\pi M^2 \left(1 + \sqrt{1 - J^2/M^4}\right) .$$

The event horizon exists only if the angular momentum is bounded by the inequality $-M^2 \leq J \leq M^2$, which in everyday terms is a very strong constraint. The exact solution also has an inner horizon with area:

$$A_- = 8\pi M^2 \left(1 - \sqrt{1 - J^2/M^4}\right) .$$

If $J/M^2 = \pm 1$ the two horizons coincide, and we have an extreme black hole with vanishing surface gravity (that is, vanishing Hawking temperature).

Outer and Inner Horizon

Since the inner horizon will play an important role below, we should perhaps say that there is no reason to believe that the spinning black hole at the center of the Milky Way has an inner horizon. The sense in which that black hole is likely to be modeled by the Kerr solution is bound up with asymptotics, just as the equilibrium states and quasi-static processes of textbook thermodynamics are useful shorthands for a more complicated reality.

We take the view that the inner horizon is an important feature of the equilibrium state. Its thermodynamics has already received some attention in the literature. Consequently we have two distinct entropy functions to study, namely:

$$S_{\pm} = S_{\pm}(M,J) = \frac{k}{4}A_{\pm} = 2M^2\left(1 \pm \sqrt{1 - J^2/M^4}\right) ,$$

where we have set Boltzmann's constant $k = 1/\pi$.

The same form of M(S,J) for outer and inner horizon

There will also be two different Hawking temperatures:

$$T_\pm = \pm \frac{1}{4M} \frac{\sqrt{1 - J^2/M^4}}{1 \pm \sqrt{1 - J^2/M^4}} \ .$$

They both vanish in the extreme limit. Otherwise, T_+ is positive, and T_- is negative.

If we invert the entropy function to obtain the mass M as a function of entropy and angular momentum, we find the same functional form in both cases,

$$M = \sqrt{\frac{S_\pm}{4} + \frac{J^2}{S_\pm}} \ .$$

Consequently, we will obtain the same expression for its Hessian, the Weinhold metric, in both cases; only the range of the coordinates will differ.

Ruppeiner line element in coordinates M and a

The expressions for the Weinhold and Ruppeiner metrics in their defining coordinates are not very illuminating. Changing to the dimensionless coordinate:

$$a = \frac{J}{M^2} \ , \qquad -1 \leq a \leq 1 \ ,$$

and using our expression for $T_\pm$, we obtain for the two Ruppeiner metrics:

$$ds_\pm^2 = \frac{1}{T_\pm}\left[-\frac{dM^2}{M} + \frac{M}{2}\frac{da^2}{(1-a^2)(1\pm\sqrt{1-a^2})}\right] \ .$$

Manifestly flat Weinhold metric

The expression within brackets gives the Weinhold metric. To bring the latter to manifestly flat form, we perform a sequence of coordinate transformations, *viz.*:

$$a = \sin 2\beta \ , \qquad \cosh 2\alpha = \frac{1}{\cos\beta} \ ,$$

$$t = 2\sqrt{M}\cosh\alpha \ , \qquad x = 2\sqrt{M}\sinh\alpha \ .$$

The result is that:

$$ds_{\pm}^2 = \frac{1}{T_{\pm}}\left[-dt^2 + dx^2\right] \ .$$

Coordinate ranges

From the coordinate transformations it is clear that the Gibbs surface for the Ruppeiner metric ds_+^2 associated with the outer horizon is a timelike wedge, with a locally flat Minkowski metric for its Weinhold metric. The wedge is bounded by:

$$-\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} \leq \frac{x}{t} \leq \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \tan\frac{\pi}{8} \ .$$

Its opening angle is 45°. The Ruppeiner metric itself is not defined on the edge of the wedge, since the conformal factor diverges there. However, the Weinhold metric can evidently be analytically extended. By increasing the coordinate range, we include also the Gibbs surface corresponding to the inner horizon. The combined Gibbs surface is isometric to the future null cone of Minkowski space, as far as its Weinhold metric is concerned. We find this satisfying.

M(t,x), S(t,x), and T(t,x)

Using the Minkowski space coordinates, we can now give unifying expressions for the thermodynamic functions. The mass and the entropy are:

$$M = \frac{t^2 - x^2}{4}$$

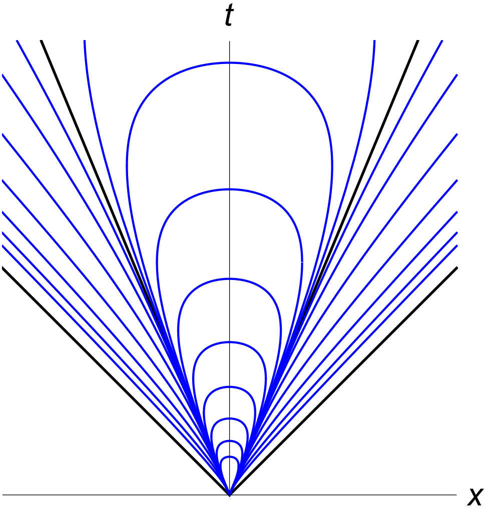
$$S = 2M^2(1 \pm \sqrt{1 - a^2}) = M^2(1 + \cos 2\beta) = \frac{(t^2 - x^2)^4}{4(t^2 + x^2)^2} \ .$$

Both of them vanish on the light cone (while they remain finite on the edge of the wedge, where the extreme black holes sit). The Hawking temperatures $T_{\pm}$ are unified to:

$$T = \frac{(t^2 - x^2 - 2tx)(t^2 - x^2 + 2tx)}{2(t^2 - x^2)^3} \ .$$

Its variation over the Gibbs surface is shown in the next slide.

Hawking Temperature for Kerr Black Hole

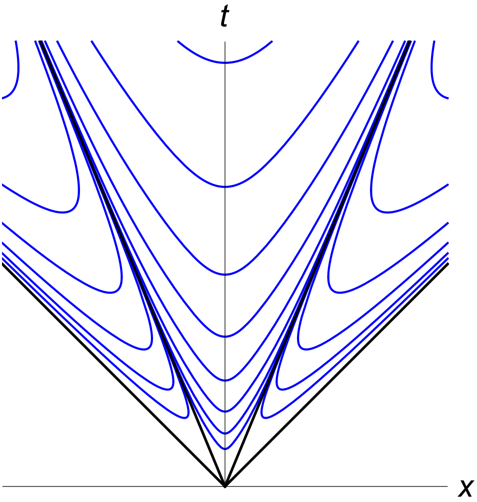


Contour curves of equal Hawking temperature

$$T = \frac{(t^2 - x^2 - 2tx)(t^2 - x^2 + 2tx)}{2(t^2 - x^2)^3} .$$

The Hawking temperature vanishes at the edge of the wedge that corresponds to the outer horizon, is negative outside and diverges on the null cone.

Ruppeiner scalar R for Kerr

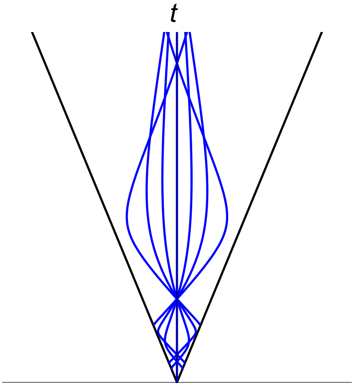


Contour curves of equal Ruppeiner scalar curvature

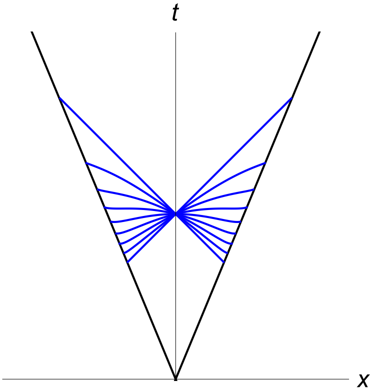
$$R = -\frac{4(t^4+10t^2x^2+x^4)}{(t^2-x^2)^2(t^4-6t^2x^2+x^4)} \ .$$

It is negative inside the wedge, positive outside and diverges both at the edge of the wedge and on the null cone.

Geodesic Curves in Ruppeiner Geometry for Kerr

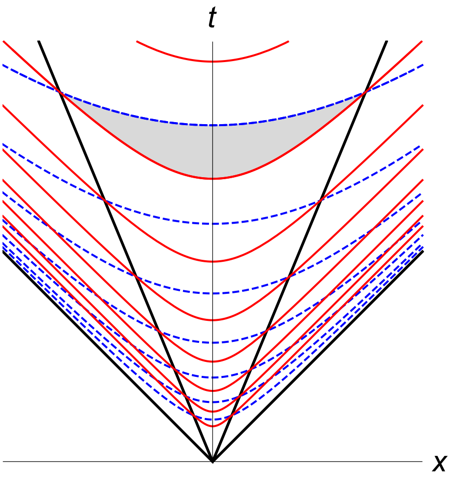


Timelike geodesics inside the wedge.



Spacelike and null geodesics inside the wedge.

Entropy and Mass for Kerr



Contour curves for entropy $S = \frac{(t^2-x^2)^4}{4(t^2+x^2)^2}$ (red) and mass $M = \frac{t^2-x^2}{4}$ (blue, dashed).

By moving inside the grey area, from near the edge of the wedge (large a) towards the center, one is able to decrease the mass (extract energy), even though the area of the event horizon (the entropy) increases as it must according to Einstein's theory.

Thermodynamic optimization of the Penrose process

This approach has been used in an attempt to find the maximal amount of energy that can be extracted from a Kerr black hole in a finite time by Bravetti, Gruber and Lopez-Monsalvo (2016).

From an extreme Kerr black hole can at most

$$1 - \frac{1}{\sqrt{2}} \approx 0.29$$

of the original mass be extracted in a Penrose process but due to losses this theoretical value can in practice not be achieved.

Thank you for listening!

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