Causality theory for $C^{1,1}$ metrics

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joint work with

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- Only a few [3] consider lower regularity metrics.
- For C^2 metrics the inverse function theorem shows that locally the causal structure is as in Minkowski space, in the sense that

$$\exp_p : \tilde{U} \to U \quad \text{is a } C^1\text{-diffeomorphism}$$

and

$$I^+(p) \cap U = \exp_p\left(I^+(0) \cap \tilde{U}\right)$$

For metrics below C^2 this result is problematic.

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(2) Physical models: Want to be able to include situations where there is a shock in the matter or a jump in the energy-momentum tensor across an interface, such as the boundary of a star.

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(3) Singularity Theorems: Failure to be C^2 need not be very singular (see below and next talk). So we want to be able to prove singularity theorems in lower regularity than C^2. Need causality theory in lower regularity than C^2.

Why $C^{1,1}$ metrics?

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Problems below $C^{1,1}$:

- Below $C^{1,1}$ geodesic convexity may no longer hold [Hartman & Wintner]
- Below $C^{0,1}$ may not be able to deform a causal curve that is not everywhere null into a timelike curve. (Pushup lemma fails) [Chruściel & Grant].
- Below $C^{0,1}$ exist "bubbling metrics" with light-cones having non-empty interior [Chruściel & Grant].

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Theorem: Let M be a smooth manifold with a $C^{1,1}$-pseudo-Riemannian metric g. Then each point $p \in M$ possesses a basis of totally normal neighbourhoods.

Remark: A totally normal set is geodesically convex.

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- Less precise information.
- Some results are valid only almost everywhere.

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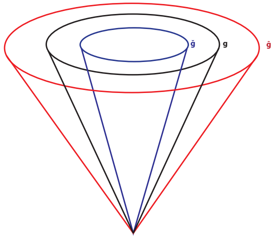
Both needed to prove singularity theorems in low regularity.

Chruściel-Grant regularisation of the metric

Sandwich null cones of g between null cones of two families of smooth metrics so that $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$.



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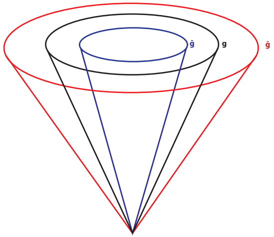

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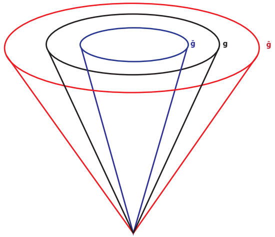

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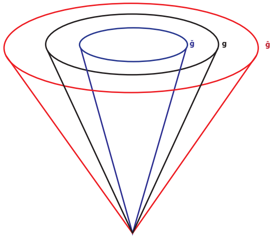

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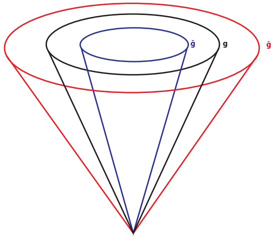

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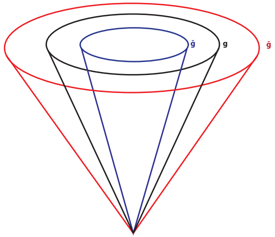

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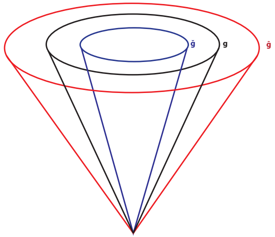

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- Glue w.r.t. x and ε to obtain Lorentzian metric $\check{g}_\varepsilon \prec g$.
- Similarly obtain $\hat{g}_\varepsilon$ with $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$.

Properties of approximating metrics

- Given a Lorentzian metric of some prescribed regularity (e.g., Sobolev, Hölder, etc.), the convergence of the inner and outer regularisations $\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ to g is as good locally as that of regularisation by convolution.
- If g is a metric of general pseudo-Riemannian signature, then we may produce regularisations $\tilde{g}_\varepsilon$ that are pseudo-Riemannian metrics on all of M of the same signature as g
- Every point has a basis of normal neighbourhoods U with $\exp_p : \tilde{U} \to U$, such that for ε sufficiently small, all $\exp_p^{g_\varepsilon}$ are diffeomorphisms with domain $\tilde{U}$. Moreover, the inverse maps $(\exp_p^{g_\varepsilon})^{-1}$ also are defined on a common neighbourhood of p for ε small, and converge locally uniformly to $\exp_p^{-1}$.

$C^{1,1}$ causality results

The following is the main local causality result for $C^{1,1}$ metrics:

Theorem: [KSSV] Let g be a $C^{1,1}$-Lorentzian metric, and let $p \in M$. Then p has a basis of normal neighbourhoods U, $\exp_p : \tilde{U} \to U$ a bi-Lipschitz homeomorphism, such that:

$$\begin{aligned} I^+(p, U) &= \exp_p(I^+(0) \cap \tilde{U}) \\ J^+(p, U) &= \exp_p(J^+(0) \cap \tilde{U}) \\ \partial I^+(p, U) = \partial J^+(p, U) &= \exp_p(\partial I^+(0) \cap \tilde{U}) \end{aligned}$$

Proof: Proof of Theorem

Remark: We follow [Chruściel] in that we base our approach to causality theory on locally Lipschitz curves. This definition differs from that of [Minguzzi] (and others) where the corresponding curves are required to be C^1. However we have the following Corollary to the above result:

Corollary: Let $U \subseteq M$ be open, $p \in U$. Then the sets $I^+(p, U)$, $J^+(p, U)$ remain unchanged if Lipschitz curves are replaced by piecewise C^1 curves, or in fact by broken geodesics.

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Proof of $J^+(p, U) = \exp_p(J^+(0) \cap \tilde{U})$

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1. We first show that $J^+(p, U) \supseteq \exp_p(J^+(0) \cap \tilde{U})$.

Let $v \in \tilde{U}$ and let $\alpha(t) = \exp_p(tv)$. Set $\alpha_\varepsilon(t) := \exp_p^{\hat{g}_\varepsilon}(tv)$.
Then $\alpha_\varepsilon \to \alpha$ in C^1.
Hence applying the smooth Gauss lemma for each ε it follows that:
$g(\alpha'(t), \alpha'(t)) = \lim_{\varepsilon \to 0} \hat{g}_\varepsilon(\alpha'_\varepsilon(t), \alpha'_\varepsilon(t)) = \lim_{\varepsilon \to 0} (\hat{g}_\varepsilon)_p(v, v) = g_p(v, v)$.
So $v \in \tilde{U} \cap J^+(0) \Rightarrow \exp_p(v) \in J^+(p, U)$.

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So $v \in \tilde{U} \cap J^+(0) \Rightarrow \exp_p(v) \in J^+(p, U)$.

2. We now show that $J^+(p, U) \subseteq \exp_p(J^+(0) \cap \tilde{U})$.

Let $\alpha : [0, 1] \to U$ be a FD causal curve in U from p
Let $\beta := (\exp_p)^{-1} \circ \alpha$ be the corresponding curve in $\tilde{U} \subset T_pM$.
Note that α is TL wrt $\hat{g}_\varepsilon$ and set $\beta_\varepsilon := (\exp_p^{\hat{g}_\varepsilon})^{-1} \circ \alpha$.
Then by smooth causality $\beta_\varepsilon([0, 1]) \subseteq I^+_{\hat{g}_\varepsilon(p)}(0)$ for all $\varepsilon < \varepsilon_0$.
Let $\tilde{Q}(v) = g_p(v, v)$ and $\tilde{Q}_\varepsilon(v) = (\hat{g}_\varepsilon)_p(v, v)$ be quadratic forms on T_pM.
Then $\beta_\varepsilon \to \beta$ uniformly, and $\tilde{Q}_\varepsilon \to \tilde{Q}$ locally uniformly.
So $\tilde{Q}(\beta(t)) = \lim_{\varepsilon \to 0} \tilde{Q}_\varepsilon(\beta_\varepsilon(t)) \leq 0$ Hence $\beta((0, 1]) \subseteq J^+(0) \cap \tilde{U}$.

Back