Elastic waves in spherically symmetric elastic spacetimes

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Summary

We consider spherically symmetric spacetimes with elastic matter and determine the propagation speed of elastic waves in the radial direction.
The propagation speed depends on the density, radial pressure and elasticity tensor components.
The local causality condition for the speed of elastic waves is analysed for shear free spherically symmetric elastic solutions of the Einstein field equations.

Outline

- Introduction to general relativistic elasticity

  (Carter and Quintana (1972), Kijowski and Magli (1992), Karlovini and Samuelsson (2003))

- Wave propagation in general relativistic elastic spacetimes

  (Karlovini and Samuelsson (2003), Carter (1973))

- Applications to spherically symmetric elastic spacetimes

  (Brito, Carot, Vaz (2010))

Configuration mapping

The space-time configuration of the material is described by the mapping

$$\Psi : M \longrightarrow X.$$

- (M, g_{ab}) space-time with coordinate system $\{x^a\}$, $a = 0, 1, 2, 3$
- (X, γ_{AB}) *material space* with *material metric* γ_{AB} and coordinate system $\{y^A\}$, $A = 1, 2, 3$

The material space is a three-dimensional manifold, whose points represent the particles of the material.

Pulled-back material metric

$$k_{ab} = \Psi^* \gamma_{AB} = y^A_a y^B_b \gamma_{AB}$$

$y^A_a = \dfrac{\partial y^A}{\partial x^a}$ is the relativistic deformation gradient.

Velocity field of the matter

The velocity field of the matter $u^a \in T_pM$ is defined by the conditions

$$u^a y^A_a = 0$$
$$u^a u_a = -1$$
$$u^0 > 0$$

Relativistic strain tensor

The operator ${K^a}_b = -u^a u_b + {k^a}_b$ can be used to measure the state of strain of the material.

The relativistic strain tensor is defined by

$$s_{ab} = \frac{1}{2}(h_{ab} - k_{ab}) = \frac{1}{2}(g_{ab} - K_{ab}),$$

where $h_{ab} = g_{ab} + u_a u_b$.

The material is in an unstrained state if $s_{ab} = 0$.

Energy-momentum tensor

The energy-momentum tensor for elastic matter

$$T_{ab} = \rho u_a u_b + p_{ab} = \rho u_a u_b + p h_{ab} + \pi_{ab}$$

can be written as

$$T^a{}_b = -\rho\, \delta^a_b + \frac{\partial \rho}{\partial I_3} \det K\, h^a{}_b - \left(\mathrm{Tr} K\, \frac{\partial \rho}{\partial I_2} - \frac{\partial \rho}{\partial I_1} \right) k^a{}_b + \frac{\partial \rho}{\partial I_2}\, k^a{}_c\, k^c{}_b.$$

- $\rho = \epsilon v$ energy density
- ϵ particle number density
- $v = v(I_1, I_2, I_3)$ constitutive equation

I_1, I_2 and I_3 are the invariants of K:

$I_1 = \frac{1}{2}\left(\mathrm{Tr} K - 4\right), \quad I_2 = \frac{1}{4}\left[\mathrm{Tr} K^2 - (\mathrm{Tr} K)^2\right] + 3, \quad I_3 = \frac{1}{2}\left(\det K - 1\right)$

Equations of motion

The conservation law

$$T^{ab}{}_{;b} = 0$$

implies the following equations of motion

- $\rho_{,c} u^c = -\rho u^c{}_{;c} - p^{cd} u_{c;d}$
- $p^{ab}{}_{;c} u^c = 2u^{(a} p^{b)c} \dot{u}_c + 2p^{c(a} u^{b)}{}_{;c} - p^{ab} u^c{}_{;c} - E^{abcd} u_{c;d}$

where

- $\dot{u}^a = u^a{}_{;c} u^c$
- E^{abcd} is the relativistic elasticity tensor.

Relativistic elasticity tensor

The relativistic elasticity tensor

$$E^{abcd} = -2\frac{\partial p^{ab}}{\partial h_{cd}} - p^{ab}h^{cd}$$

satisfies the symmetry conditions

$$E^{abcd} = E^{(ab)(cd)} = E^{cdab}$$

and is orthogonal to the velocity of the flow

$$E^{abcd}u_d = 0.$$

It can be rewritten as

$$E^{abcd} = 4\epsilon\frac{\partial^2 v}{\partial g_{ab} g_{cd}}.$$

Relativistic Hadamard elasticity tensor

The relativistic Hadamard elasticity tensor is defined by

$$A^{abcd} = E^{abcd} - h^{ac} p^{bd}.$$

This tensor has the symmetry

$$A^{abcd} = A^{cdab}$$

and is orthogonal to the velocity of the flow

$$A^{abcd} u_d = 0.$$

Sound wave front

The sound wave front is a hypersurface which moves in spatial direction ν^a with speed w with respect to the flow u^a at some point.
The normal to the wave front lies in the direction of the vector

$$\lambda_a = \nu_a - w u_a.$$

The propagation direction vector ν^a satisfies

$$\nu^a \nu_a = 1, \;\; \nu^a u_a = 0.$$

The speed of propagation of the wave front

$$w = \lambda^a u_a$$

must satisfy

$$w^2 \leq 1.$$

Sound wave front

The acceleration vector $\dot{u}^a$ can have a jump discontinuity across the hypersurface

$$[\dot{u}^a] = \alpha \iota^a.$$

- α is the amplitude of the wave front.
- ι^a is the *polarization vector* of the wave front, satisfying

$$\iota^a \iota_a = 1$$

$$\iota^a u_a = 0.$$

Sound wave front

From the conservation law, one obtains

$$\{w^2(\rho h^{ac} + p^{ac}) - Q^{ac}\}\iota_c = 0.$$

Q^{ac} is the relativistic Fresnel tensor defined by

$$\begin{aligned} Q^{ac} &= A^{abcd} v_b v_d \\ &= (E^{abcd} - h^{ac} p^{bd}) v_b v_d, \end{aligned}$$

satisfying

- $Q^{ac} = Q^{(ac)}$
- $Q^{ac} u_c = 0.$

Spherically symmetric space-time configuration

- Space-time metric g

$$ds^2 = -a^2(t,r)dt^2 + b^2(t,r)dr^2 + Y^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2),$$

$$g_{ab} = -u_a u_b + e_{1a}e_{1b} + e_{2a}e_{2b} + e_{3a}e_{3b}$$

$$e_{1a} = (0, b, 0, 0)\,,\ e_{2a} = (0, 0, Y, 0),\ e_{3a} = (0, 0, 0, Y\sin\theta)$$

- Velocity vector of the flow

$$u^a = \left(a^{-1}, 0, 0, 0\right)$$

- Pulled-back material metric k

$$d\Sigma^2 = f^2(r)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

Spherically symmetric space-time configuration

- The operator $K^a{}_b = -u^a u_b + k^a{}_b$ is given by

$$K^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{f^2(r)}{b^2} & 0 & 0 \\ 0 & 0 & f^2(r)\,\frac{r^2}{Y^2} & 0 \\ 0 & 0 & 0 & f^2(r)\,\frac{r^2}{Y^2} \end{pmatrix}.$$

  It has one eigenvalue equal to 1 and the other eigenvalues are $\eta = \frac{f^2(r)}{b^2}$ and $s = f^2(r)\,\frac{r^2}{Y^2}$.

- Invariants of K:

$$I_1 = \frac{1}{2}\left(\eta + 2s - 3\right)$$
$$I_2 = -\frac{1}{2}\left(s^2 + 2\eta s + \eta + 2s\right) - 3$$
$$I_3 = \frac{1}{2}\left(\eta s^2 - 1\right)$$

Energy-momentum tensor

$$T_0^0 = -\epsilon v,$$
$$T_1^1 = 2\,\epsilon\,\eta\,\frac{\partial v}{\partial \eta} = p_1,$$
$$T_2^2 = \epsilon\,s\,\frac{\partial v}{\partial s} = p_2.$$

- Constitutive equation $v = v(s, \eta)$
- Energy density $\rho = \epsilon v = \epsilon_0\, s\,\sqrt{\eta}\, v(s, \eta)$

Elasticity tensor

$$
\begin{aligned}
E^{abcd} &= 4\epsilon \frac{\partial^2 v}{\partial g_{ab} \partial g_{cd}} \\
&= 4\epsilon \left[\left(2\eta \frac{\partial v}{\partial \eta} + \eta^2 \frac{\partial^2 v}{\partial \eta^2} \right) \frac{1}{b^4} \delta^a_r \delta^b_r \delta^c_r \delta^d_r \right. \\
&+ \left(\frac{2\eta^2}{\eta - s} \frac{\partial v}{\partial \eta} + \frac{s^2}{s - \eta} \frac{\partial v}{\partial s} \right) \left(\frac{\delta^{(a}_r \delta^{b)}_\theta \delta^{(c}_r \delta^{d)}_\theta}{b^2 Y^2} + \frac{\delta^{(a}_r \delta^{b)}_\phi \delta^{(c}_r \delta^{d)}_\phi}{b^2 Y^2 \sin^2\theta} \right) \\
&+ \frac{1}{2} \eta s \frac{\partial^2 v}{\partial \eta \partial s} \left(\frac{\delta^a_r \delta^b_r \delta^c_\theta \delta^d_\theta + \delta^c_r \delta^d_r \delta^a_\theta \delta^b_\theta}{b^2 Y^2} + \frac{\delta^a_r \delta^b_r \delta^c_{\phi 0} \delta^d_\phi + \delta^c_r \delta^d_r \delta^a_\phi \delta^b_\phi}{b^2 Y^2 \sin^2\theta} \right) \\
&+ \left(\frac{3}{4} s \frac{\partial v}{\partial s} + \frac{1}{4} s^2 \frac{\partial^2 v}{\partial s^2} \right) \left(\frac{\delta^a_\theta \delta^b_\theta \delta^c_\theta \delta^d_\theta}{Y^4} + \frac{\delta^a_\phi \delta^b_\phi \delta^c_\phi \delta^d_\phi}{Y^4 \sin^4\theta} \right) \\
&+ \left. \left(\frac{1}{4} s \frac{\partial v}{\partial s} + \frac{1}{4} s^2 \frac{\partial^2 v}{\partial s^2} \right) \frac{\delta^a_\theta \delta^b_\theta \delta^c_\phi \delta^d_\phi + \delta^a_\phi \delta^b_\phi \delta^c_\theta \delta^d_\theta}{Y^4 \sin^2\theta} \right]
\end{aligned}
$$

Propagation direction vector

For longitudinal waves in the radial direction:

$$\nu^a = \iota^a = \left[0, b^{-1}, 0, 0\right]$$

Propagation speed

From the characteristic equation

$$\{w^2(\rho h^{ac} + p^{ac}) - Q^{ac}\}\iota_c = 0,$$

one obtains

$$w^2 = \frac{bQ^{rr}}{\frac{\rho}{b} + bp^{rr}} = \frac{b^3 E^{rrrr} - bp^{rr}}{\frac{\rho}{b} + bp^{rr}}.$$

Propagation speed

Since

$$E^{rrrr} = \frac{4\epsilon}{b^4}\left(2\eta\frac{\partial v}{\partial \eta} + \eta^2\frac{\partial^2 v}{\partial \eta^2}\right)$$

and

$$\rho = \epsilon v, \quad p^{rr} = T^{rr} = 2\epsilon\eta\frac{\partial v}{\partial \eta}\frac{1}{b^2},$$

then

$$w^2 = \frac{6\epsilon\eta\frac{\partial v}{\partial \eta} + 4\epsilon\eta^2\frac{\partial^2 v}{\partial \eta^2}}{\epsilon v + 2\epsilon\eta\frac{\partial v}{\partial \eta}}$$

$$= \frac{3p_1 + 4\epsilon\eta^2\frac{\partial^2 v}{\partial \eta^2}}{\rho + p_1}.$$

Examples

Static shear-free solution

- Space-time metric

$$ds^2 = -e^{10r^2} dt^2 + e^{-5r^2} \left(dr^2 + d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Pulled-back material metric

$$d\Sigma^2 = f^2(r)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \quad f(r) = \frac{e^{-\frac{5}{2}r^2}}{(75r^2+1)^{\frac{1}{3}}}$$

- Energy density

$$\rho = \epsilon v = \frac{1}{8\pi} e^{5r^2}(11 - 25r^2)$$

- Radial pressure and tangential pressure

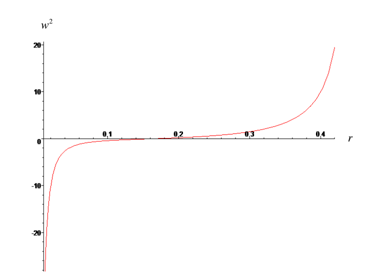
$$p_1 = 2\epsilon\eta\frac{\partial v}{\partial \eta} = -\frac{1}{8\pi}e^{5r^2}(25r^2+1), \quad p_2 = \epsilon s\frac{\partial v}{\partial s} = \frac{1}{8\pi}25r^2 e^{5r^2}$$

Examples

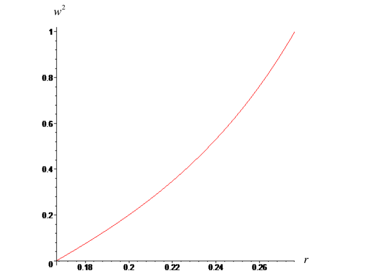
Static shear-free solution

- Propagation speed

$$w^2 = \frac{9375r^6 + 1750r^4 - 20r^2 - 1}{-1250r^4 + 250r^2}$$



$r \in \left(0, \frac{1}{\sqrt{5}}\right)$



$r \in (0.166797, 0.276078)$

Examples

Non-static shear-free solution

- Space-time metric

$$ds^2 = -\,dt^2 + t^2 B^2(r)\left(dr^2 + d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$B(r) = \frac{\sqrt{3}}{9}\left(2 + 3(r - r_0)^2\right)^{\frac{3}{2}}$$

- Pulled-back material metric

$$d\Sigma^2 = f^2(r)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

$$f(r) = \exp\left\{\frac{-15-9(r-r_0)^2+\frac{3}{2}(2+3(r-r_0)^2)\sqrt{4+6(r-r_0)^2}\tanh^{-1}\left(\frac{2}{\sqrt{4+6(r-r_0)^2}}\right)}{\sqrt{6+9\,(r-r_0)^2}(2+3\,(r-r_0)^2)}\right\}$$

Examples

Non-static shear-free solution

- Energy density

$$\rho = -T^0_0 = \frac{1}{8\pi\, t^2}\left(-\frac{2B''}{B^3} + \frac{B'^2}{B^4} + \frac{1}{B^2} + 3\right)$$

- Radial pressure

$$p_1 = T^1_1 = \frac{1}{8\pi\, t^2}\left(\frac{B'^2}{B^4} - \frac{1}{B^2} - 1\right)$$

- Tangential pressure

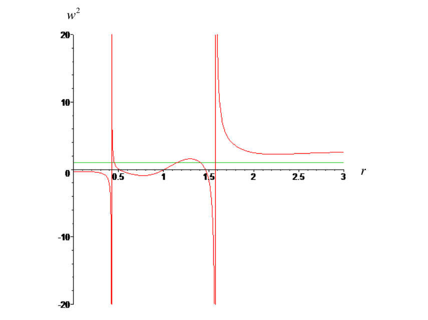
$$p_2 = T^2_2 = \frac{1}{8\pi\, t^2}\left(\frac{B''}{B^3} - \frac{B'^2}{B^4} - 1\right)$$

Examples

Non-static shear-free solution

- Propagation speed

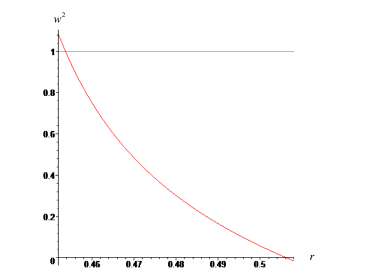
$$w^2 = \frac{\left(\frac{2}{3}-B\right)\left(-\frac{B'^2}{B}+B^3+B\right)+rB'\left(-B''+\frac{B'^2}{B^2}+\frac{B'^2}{B}+B^3-B^2-1\right)}{-BB''+B'^2+B^4}$$



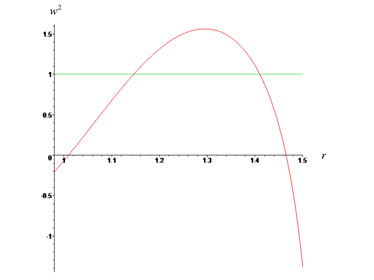
w^2 for $r_0 = 1$ and $r \in (0, 3)$

Examples

Non-static shear-free solution



$0 \leq w^2 \leq 1$ for $r \in (0.453677, 0.506165)$



$0 \leq w^2 \leq 1$ for $r \in (1.009721, 1.145846)$ and for $r \in (1.410349, 1.465197)$

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