Cosmological transition amplitudes with Proper Vertex

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The Lorentzian proper vertex amplitude: Asymptotics *1505.06683*
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Outline

Motivation

- Introduce triangulation Δ and its dual Δ^* for the manifold M. Ponzano-Regge model for $3D$ gravity

$$Z(\Delta) = \sum_{j_e} \prod_{e\in\Delta} (-1)^{2j_e}(2j_e+1) \prod_{t\in\Delta} (-1)^{j_1+j_2+j_3} \prod_{\tau\in\Delta} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

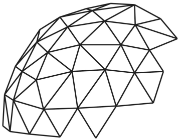
- The spinfoam theory is motivated by the Ponzano-Regge model and defined by the vertex amplitude A_v in the partition function:

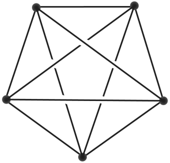
$$Z(\Delta) = \sum_{j} \prod_{f\in\Delta^*} A_f \prod_{v\in\Delta^*} A_v$$

- To study the semi-classical limit of the theory want to take the limit of large spins.
- The behavior of the amplitude for large spins can be studied using stationary phase methods if the vertex amplitude is written as:

$$A_v = \int e^S$$

Coherent states





- Boundary space is parametrized by $SU(2)$ spin network functions labelled by spin j_{ab} and two vectors ψ_{ab}, ψ_{ba} in corresponding irrep of $SU(2)$
- Choose these vectors to be coherent states corresponding to spinors ξ_{ab}, ξ_{ba} (or equivalently, normals $\vec{n}_{ab}$, $\vec{n}_{ba}$)
- This choice gives rise to Livine-Speziale coherent states $C_{\xi_{ab}}$ peaked on intrinsic geometry of the boundary.

Euclidean EPRL vertex amplitude

- EPRL vertex amplitude for coherent boundary states ($\gamma < 1$)

$$A_v \sim \int_{G \in Spin(4)^5} \left(\prod_a dG_a \right) \prod_{a<b} \langle J \iota_{j_{ab}}^{s_{ab}^- s_{ab}^+} n_{ab}, \rho(G_{ab})^{-1} \iota_{j_{ab}}^{s_{ab}^- s_{ab}^+} n_{ba} \rangle$$

(with $s_{ab}^{\pm} = \frac{1}{2}|1+\gamma| j_{ab}, G_a = (X^-, X^+)$ and $G_{ab} = (G_a)^{-1} G_b$) which can be rewritten as

$$A_v \sim \int_{G \in Spin(4)^5} \left(\prod_a dG_a \right) e^S$$

- Here

$$S = \sum_{a<b} \sum_{\pm} 2 j_{ab}^{\pm} \log \langle J n_{ab} | X_{ab}^{\pm} | n_{ba} \rangle$$

- In this form the semi-classical limit of the amplitude can be analyzed using stationary phase methods.

Asymptotics of Euclidean EPRL vertex

- The asymptotic analysis was performed by Barrett, Dowdall, Hellmann, Fairbairn, Gomes (2009).
- In particular, they found that for boundary Regge geometry of Euclidean 4-simplex with spins λj_{ab} in the limit $\lambda \to \infty$

$$A_v \sim \frac{1}{\lambda^{12}} \left(2N_{+-} \cos\left(\lambda\gamma \sum_{a<b} j_{ab}\Theta_{ab} \right) + N_{++} \exp\left(i\lambda \sum_{a<b} j_{ab}\Theta_{ab} \right) + N_{--} \exp\left(-i\lambda \sum_{a<b} j_{ab}\Theta_{ab} \right) \right)$$

where Θ_{ab} are dihedral angles in the boundary Euclidean 4-simplex, γ is the Barbero-Immirzi parameter and factors N_{+-}, N_{++}, N_{--} are independent of λ and given in Barrett *et al*.

Lorentzian EPRL vertex amplitude

- EPRL vertex amplitude for coherent boundary states

$$A_v \sim \int_{g \in SL(2,\mathbb{C})^5} \delta(g_0) \left(\prod_a dg_a \right) \prod_{a<b} \langle C_{J\xi_{ab}} | \bar{g}_a^{-1} \bar{g}_b | C_{\xi_{ba}} \rangle$$

which can be rewritten as

$$A_v \sim \int_{g \in SL(2,\mathbb{C})^5} \delta(g_0) \left(\prod_a dg_a \right) \int_{z \in (\mathbb{CP}^1)^{10}} \prod_{a<b} d\mu_{z_{ab}} e^S$$

- Here

$$S = \sum_{a<b} \left(j_{ab} \log \frac{\langle J\xi_{ab}, Z_{ab} \rangle^2 \langle Z_{ba}, \xi_{ba} \rangle^2}{\langle Z_{ab}, Z_{ab} \rangle \langle Z_{ba}, Z_{ba} \rangle} + i\gamma j_{ab} \log \frac{\langle Z_{ba}, Z_{ba} \rangle}{\langle Z_{ab}, Z_{ab} \rangle} \right)$$

with $Z_{ab} = g_a^\dagger z_{ab}$, $Z_{ba} = g_b^\dagger z_{ab}$.

- In this form the semi-classical limit of the amplitude can be analyzed using stationary phase methods (similarly to Euclidean case).

Asymptotics of Lorentzian EPRL vertex

- The asymptotic analysis was performed by Barrett, Dowdall, Hellmann, Fairbairn, Pereira (2010).
- For boundary Regge geometry of Lorentzian 4-simplex with spins λj_{ab} in the limit $\lambda \to \infty$

$$A_v \sim \frac{1}{\lambda^{12}} \left(N_+ \exp\left(i\lambda\gamma \sum_{a<b} j_{ab}\Theta_{ab} \right) + N_- \exp\left(-i\lambda\gamma \sum_{a<b} j_{ab}\Theta_{ab} \right) \right)$$

  where Θ_{ab} are dihedral angles in the boundary Lorentzian 4-simplex.
- For boundary Regge geometry of Euclidean 4-simplex with spins λj_{ab} in the limit $\lambda \to \infty$

$$A_v \sim \frac{1}{\lambda^{12}} \left(N_+^E \exp\left(i\lambda \sum_{a<b} j_{ab}\Theta^E_{ab} \right) + N_-^E \exp\left(-i\lambda \sum_{a<b} j_{ab}\Theta^E_{ab} \right) \right)$$

  where Θ^E_{ab} are dihedral angles in the boundary Euclidean 4-simplex.
- The presence of two terms with differing signs in exponent creates problems, especially for the case of multiple 4-simplices.

Einstein-Hilbert sector

- In spin-foams quantize gravity as BF-theory, impose constraints on B.

$$S[B,\omega] = \frac{1}{\kappa} \int \left(*B^{IJ} + \frac{1}{\gamma} B^{IJ} \right) \wedge F_{IJ}(\omega)$$

- The solutions to constraints fall into 5 sectors:
  - (I$\pm$) $B = \pm(e \wedge e)$
  - (II$\pm$) $B = \pm * (e \wedge e)$
  - (deg) $tr(*B \wedge B) = 0$
- Only (II$\pm$) correspond to gravity. We call it the **Einstein-Hilbert sector**.

Euclidean Proper Vertex amplitude

- Restrict to the Einstein-Hilbert sector by introducing a projector into the vertex amplitude (Engle *1111.2865*)

$$A_v^{(+)} \sim \int_{G \in Spin(4)^5} \left(\prod_a dG_a \right) \prod_{a<b} \langle J \iota_{j_{ab}}^{s_{ab}^- s_{ab}^+} n_{ab}, \rho(G_{ab}) \iota_{j_{ab}}^{s_{ab}^- s_{ab}^+} \Pi_{ba}(\{G_{a'b'}\} n_{ba} \rangle$$

- Here the projector is defined as

$$\Pi_{ba}(\{G_{a'b'}\}) := \Pi_{(0,\infty)}(\beta_{ab}(\{G_{a'}\}) tr(X_{ba}^- X_{ab}^+ \sigma_i) \hat{\mathbf{L}}^i)$$

- Projector is discontinuous when $X_{ab}^\pm \in SU(2)$.

Lorentzian Proper Vertex amplitude

- Restrict to the Einstein-Hilbert sector by introducing a projector into the vertex amplitude (*1502.04640* by Engle, Zipfel)

$$A_v^{(+)} \sim \int_{g \in SL(2,\mathbb{C})^5} \delta(g_0) \left(\prod_a dg_a \right) \prod_{a<b} \langle C_{J\xi_{ab}} | g_a^{-1} g_b \Pi_{ba}(\{g_{ab}\}) | C_{\xi_{ba}} \rangle$$

  with $g_{ab} = g_a^{-1} g_b$.
- Rewrite as:

$$A_v^{(+)} \sim \int \delta(g_0) \left(\prod_a dg_a \right) \int \prod_{a<b} d\tilde{\mu}_{\eta_{ab}} d\mu_{z_{ab}} e^{S_{\mathrm{mEPRL}} + S_\Pi}$$

$$S_{\mathrm{mEPRL}} = \sum_{a<b} \left(j_{ab} \log \frac{\langle J\eta_{ba}, Z_{ba} \rangle^2 \langle Z_{ab}, \xi_{ab} \rangle^2}{\langle Z_{ab}, Z_{ab} \rangle \langle Z_{ba}, Z_{ba} \rangle} + i\gamma j_{ab} \log \frac{\langle Z_{ba}, Z_{ba} \rangle}{\langle Z_{ab}, Z_{ab} \rangle} \right)$$

$$S_\Pi = \sum_{a<b} \log(C_{\eta_{ba}}, \Pi_{ba}(\{g_{ab}\}) C_{\xi_{ba}})_{j_{ab}}$$

Projector

- To define projector Π_{ba} introduce normal 3-vector $\mathbf{n}_{\nu_{ba}}$:

$$n^i_{\nu_{ba}} = \beta_{ab}(\{g_{a'}\}) \frac{\mathrm{tr}(g_{ba} g^\dagger_{ba} \sigma^i)}{|\mathrm{tr}(g_{ba} g^\dagger_{ba} \sigma^i)|}$$

- To $\mathbf{n}_{\nu_{ba}}$ can associate spinor ν_{ba} such that:

$$n^i_{\nu_{ba}} = \frac{\langle \nu_{ba} | \sigma^i | \nu_{ba} \rangle}{\langle \nu_{ba} | \nu_{ba} \rangle}$$

- Projector $\Pi_{ba}(\{g_{ab}\})$ is then defined as:

$$\Pi_{ba}(\{g_{ab}\}) = \Pi_{(0,\infty)}(\mathbf{n}_{\nu_{ba}(g)} \cdot \hat{\mathbf{L}})$$

- Define

$$x_{ba} = \langle \eta_{ba}, \nu_{ba} \rangle \langle \nu_{ba}, \xi_{ba} \rangle$$

$$y_{ba} = \langle \eta_{ba}, J\nu_{ba} \rangle \langle J\nu_{ba}, \xi_{ba} \rangle$$

Asymptotic analysis

- Use stationary phase methods
- But S_Π non-linear in spins j, need to extend stationary phase analysis to non-linear actions
- Works because

$$exp(S_\Pi(\lambda, x)) \sim \tilde{B}(\lambda)\tilde{\mu}(x)exp(\lambda\tilde{S}_\Pi(x))$$

- Asymptotic form of the action:

$$\tilde{S}^\Pi_{ab} := \begin{cases} 2j_{ab}\log(x_{ab}+y_{ab}), & |x_{ab}| > |y_{ab}| \text{ and } |x_{ab}+y_{ab}|^2 \geq |4x_{ab}y_{ab}| \\ j_{ab}\log(4x_{ab}y_{ab}), & |x_{ab}| \leq |y_{ab}| \text{ or } |x_{ab}+y_{ab}|^2 < |4x_{ab}y_{ab}| \end{cases}$$

- Projector $\Pi_{ba}(\{g_{ab}\})$ is discontinuous for g_{ab} in $SU(2)$.
- Use partitions of unity to analyse contributions from different parts of integration domain.

Asymptotics of the Proper Vertex

- For Euclidean proper vertex with Regge boundary data

$$A_v^{(+)} \sim \left(\frac{1}{\lambda}\right)^{12} N_{+-} \exp\left(i\lambda\gamma \sum_{a<b} j_{ab}\Theta_{ab}\right)$$

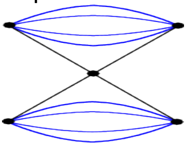
- For Lorentzian proper vertex with Regge boundary data gluing into a Lorentzian 4-simplex

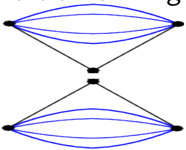
$$A_v^{(+)} \sim \left(\frac{1}{\lambda}\right)^{12} N_{+} \exp\left(i\lambda\gamma \sum_{a<b} j_{ab}\Theta_{ab}\right)$$

  while asymptotics with boundary data matching to a Euclidean 4-simplex is sub-dominant (it corresponds to the degenerate sector).
- Only one term in asymptotics.

Dipole cosmology

- Bianchi, Rovelli, Vidotto (2010) use the dipole graph to calculate a cosmological transition amplitude and find that at first order in the vertex expansion it decomposes into *Hartle-Hawking no-boundary states*





- Use the heat-kernel coherent state as the boundary state

$$\Psi_{H_l}(U_l) = \int dg_n \prod K_t(g_{s(l)}^{-1} U_l g_{t(l)} H_l^{-1})$$

with heat kernel $K_t(U) = \sum_j (2j+1) e^{-tj(j+1)} Tr(D^j(U))$

- Connection with cosmological parameters c and p is through

$$H_l = n_l e^{-iz\sigma_3/2} n_l^{-1}$$

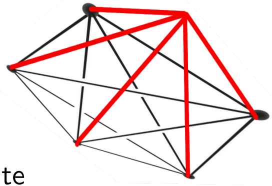
where $z = \xi + i\eta$, $\xi = \alpha c$ and $\eta = \beta p$.

Hartle-Hawking state

- We use 4-simplex graph and work with Livine-Speziale coherent states in the limit $\eta = Im(z) \to \infty$.

$$|\Psi_0\rangle = \sum_{j_{ab}} \prod_{ab} (2j+1) \exp(-\frac{(j-j^0)^2}{2\sigma^2}) \exp(-\frac{z^2\sigma^2}{2}) |j_{ab}, \Phi_a(\vec{n})\rangle$$

with $j^0 = -\frac{iz}{2t}$ and $\sigma = \frac{1}{\sqrt{2t}}$.



- Evaluate Hartle-Hawking state

$$W(z) = \langle W^{(+)}|\Psi_0\rangle = \sum_{j_{ab}} \psi(j) \langle W^{(+)}|j_{ab}, \Phi_a(\vec{n})\rangle$$

Euclidean amplitude

- To perform the calculation we use the large-volume limit identified with the large-spin limit on the graph.

- This formulation allows us to use the asymptotic behavior of the vertex to evaluate $W(z)$ (assuming Euclidean proper vertex is asymptotically linear in spins to apply stationary-phase methods).

- In the Euclidean case we get as expected a holomorphic function of z

$$W(z) = N_0 z^{-2} e^{-\frac{z^2 \sigma^2}{2}}$$

Lorentzian amplitude

- In the Lorentzian case we have to deal with critical points in the degenerate sector.
- Assuming that the integration manifold has dimension m, we show that when there is a n-dimensional submanifold of critical points where the critical point equations are only satisfied when approached from specific critical directions (since the action is not smooth) the contribution from such critical points is sub-dominant, viz. it goes as

$$I(\lambda) = o(\lambda^{-\frac{m-n+s}{2}})$$

(where s is the reduction in the number of directions of approach).

- Therefore, in Lorentzian case the amplitude is sub-dominant.

Summary

- To properly match Regge calculus in the semi-classical limit, modify the vertex amplitude introducing the projector to restrict to the Einstein-Hilbert sector.
- The semi-classical limit of the proper vertex amplitude can be analyzed using the stationary phase method.
- Asymptotic form of the proper vertex amplitude is linear in spins.
- Euclidean and Lorentzian amplitudes calculated using proper vertex differ in their scaling.
- Future work: consider multiple 4-simplices.

Outlook

- Many open questions remain in spinfoam approach.
- But New York wasn't built in a day.